K-Triviality of Closed Sets and Continuous Functions

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Abstract

We investigate the notion of K-triviality for closed sets and continuous functions in $2^{\mathbb{N}}$. For every K-trivial degreee $\mathbf{d}$, there exists a closed set of degree $\mathbf{d}$ and a continuous function of degree $\mathbf{d}$. Every K-trivial closed set contains a K-trivial real. There exists a K-trivial Π^0_1 class with no computable elements. A closed set is K-trivial if and only if it is the set of zeroes of some K-trivial continuous function. We give a density result for the Medvedev degrees of K-trivial Π^0_1 sets. If $W \leq_T A'$, then W can compute a path through every A'-decidable random closed set if and only if $W \equiv_T A'$.

1 Introduction

The study of algorithmic randomness has been an active area of research in recent years. The basic problem is to quantify the randomness of a single real

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number. Here we think of a real $r \in [0,1]$ as an infinite sequence of 0's and 1's, i.e as an element in $2^{\mathbb{N}}$. There are three basic approaches to algorithmic randomness: the measure theoretic, the compressibility and the betting approaches. All three approaches have been shown to yield the same notion of (algorithmic) randomness, under suitable effectivity assumptions. Here we will only use notions from the compressibility approach, incorporating a number of non-trivial results in this area. For background and history of algorithmic randomness we refer to [13, 12, 16].

Prefix-free (Chaitin) complexity for reals is defined as follows. Let M be a prefix-free function with domain $\subset \{0,1\}^*$. For any finite string τ, let $K_M(\tau) = min\{|\sigma| : M(\sigma) = \tau\}$. There is a *universal* prefix-free function U such that, for any prefix-free M, there is a constant c such that for all τ

$$K_U(\tau) \leq K_M(\tau) + c.$$

We let $K(\sigma) = K_U(\sigma)$. Then $x \in 2^{\mathbb{N}}$ is said to be *random* if there is a constant c such that $K(x\lceil n) \geq n - c$ for all n. This means a real x is random exactly when its initial segments are not *compressible*.

In a series of recent papers [2, 6, 3, 5], P. Brodhead, S. Dashti and the authors have defined the notion of (algorithmic) randomness for closed sets and continuous functions on $2^{\mathbb{N}}$. Some definitions are needed. For a finite string $\sigma \in \{0,1\}^n$, let $|\sigma| = n$ denote the length of n. To avoid confusion, we let $card(A)$ denote the cardinality of the set A. For two strings σ, τ, say that τ *extends* σ and write $\sigma \prec \tau$ if $|\sigma| \leq |\tau|$ and $\sigma(i) = \tau(i)$ for $i < |\sigma|$. For $x \in 2^{\mathbb{N}}$, $\sigma \prec x$ means that $\sigma(i) = x(i)$ for $i < |\sigma|$. Let $\sigma^\frown\tau$ denote the concatenation of σ and τ and let $\sigma^\frown i$ denote $\sigma^\frown(i)$ for $i = 0, 1$. Let $x\lceil n = (x(0), \ldots, x(n-1))$. Two reals x and y may be coded together into $z = x \oplus y$, where $z(2n) = x(n)$ and $z(2n+1) = y(n)$ for all n. For a finite string σ, let $I(\sigma)$ denote $\{x \in 2^{\mathbb{N}} : \sigma \prec x\}$. We shall call $I(\sigma)$ the *interval* determined by σ. Each such interval is a clopen set and the clopen sets are just finite unions of intervals. Now a nonempty closed set P may be identified with a tree $T_P \subseteq \{0,1\}^*$ where $T_P = \{\sigma : P \cap I(\sigma) \neq \emptyset\}$. Note that T_P has no dead ends. That is, if $\sigma \in T_P$, then either $\sigma^\frown 0 \in T_P$ or $\sigma^\frown 1 \in T_P$ (or both). For an arbitrary tree $T \subseteq \{0,1\}^*$, let $[T]$ denote the set of infinite paths through T. It is well-known that $P \subseteq 2^{\mathbb{N}}$ is a closed set if and only if $P = [T]$ for some tree T. P is a Π^0_1 class, or an effectively closed set, if $P = [T]$ for some computable tree T. P is a strong Π^0_2 class, or a Π^0_2 closed set, if $P = [T]$ for some Δ^0_2 tree. The complement of a Π^0_1 class is sometimes called a c.e. open set. We remark that if P is a Π^0_1 class, then T_P is a Π^0_1 set, but it is not, in general, computable. An important example of a Π^0_1 class is the class $S(A,B)$ of separating sets of disjoint c. e. sets, that is,

$$S(A,B) = \{x \in 2^{\mathbb{N}} : (\forall n)[(n \in A \rightarrow x(n) = 1) \ \wedge \ (n \in B \rightarrow x(n) = 0)]\}.$$

a natural effective enumeration $P_0, P_1, \ldots$ of the Π^0_1 For a detailed development of Π^0_1 classes, see [7].

We define a measure μ^* on the space $\mathcal{C}$ of closed subsets of $2^{\mathbb{N}}$ as follows. Given a closed set $Q \subseteq 2^{\mathbb{N}}$, let $T = T_Q$ be the tree without dead ends such that

$Q = [T]$. Let $\sigma_0, \sigma_1, \ldots$ enumerate the elements of T in order, first by length and then lexicographically. We then define the *(canonical) code* $x = x_Q = x_T$ of Q by recursion such that for each n, $x(n) = 2$ if both $\sigma_n{}^\frown 0$ and $\sigma_n{}^\frown 1$ are in T, $x(n) = 1$ if $\sigma_n{}^\frown 0 \notin T$ and $\sigma_n{}^\frown 1 \in T$, and $x(n) = 0$ if $\sigma_n{}^\frown 0 \in T$ and $\sigma_n{}^\frown 1 \notin T$. We then define μ^* by setting

$$\mu^*(\mathcal{X}) = \mu(\{x_Q : Q \in \mathcal{X}\}) \tag{1}$$

for any $\mathcal{X} \subseteq \mathcal{C}$ and μ is the standard measure on $\{0,1,2\}^{\mathbb{N}}$. Informally this means that given $\sigma \in T_Q$, there is probability $\frac{1}{3}$ that both $\sigma^\frown 0 \in T_Q$ and $\sigma^\frown 1 \in T_Q$ and, for $i = 0, 1$, there is probability $\frac{1}{3}$ that only $\sigma^\frown i \in T_Q$. In particular, this means that $Q \cap I(\sigma) \neq \emptyset$ implies that for $i = 0, 1$, $Q \cap I(\sigma^\frown i) \neq \emptyset$ with probability $\frac{2}{3}$. That is, one conceives of constructing a "random" tree T with no dead ends by starting with the empty string and then, for each node $\sigma \in T$, including either one or both of the immediate successors $\sigma^\frown i$ of σ with equal probability $\frac{1}{3}$. Brodhead, Cenzer, and Dashti [6] defined a closed set $Q \subseteq 2^{\mathbb{N}}$ to be (Martin-Löf) random if x_Q is (Martin-Löf) random. Note that the equal probability of $\frac{1}{3}$ for the three cases of branching allows the application of Schnorr's theorem that Martin-Löf randomness is equivalent to prefix-free Kolmogorov randomness. Then in [6, 3], the following results are proved. Every random closed set is perfect and contains no computable elements (in fact, it contains no n-c.e. elements). Every random closed set has measure 0 and has box dimension $\log_2 \frac{4}{3}$.

A continuous function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ may be represented by a function $f : \{0,1\}^* \to \{0,1\}^*$ such that the following hold, for all $\sigma \in \{0,1\}^*$.

(1) $|f(\sigma)| \leq |\sigma|$.

(2) $\sigma_1 \prec \sigma_2$ implies $f(\sigma_1) \prec f(\sigma_2)$.

(3) For every n, there exists m such that for all $\sigma \in \{0,1\}^m$, $|f(\sigma)| \geq n$.

(4) For all $x \in 2^{\mathbb{N}}$, $F(x) = \bigcup_n f(x \lceil n)$.

We define the space $\mathcal{F}$ of representing functions $f : \{0,1\}^* \to \{0,1\}^*$ to be those which satisfy clauses (1) and (2) above. Each $f \in \mathcal{F}$ may in turn be coded by (infinitely many) $r \in \{0,1,2\}^{\mathbb{N}}$, as follows. Enumerate $\{0,1\}^*$ in order, first by length and then lexicographically, as $\sigma_0, \sigma_1, \ldots$. Thus $\sigma_0 = \emptyset$, $\sigma_1 = (0), \sigma_2 = (1), \sigma_3 = (00), \ldots$. Then $r \in \{0,1,2\}^{\mathbb{N}}$ corresponds to the function $f_r : \{0,1\}^* \to \{0,1\}^*$ defined by declaring that $f_r(\emptyset) = \emptyset$ and that, for any σ_n with $|\sigma_n| \geq 1$,

$$f_r(\sigma_n) = \begin{cases} f_r(\sigma_k), & \text{if } r(n) = 2; \\ f_r(\sigma_k)^\frown i, & \text{if } r(n) = i < 2. \end{cases}$$

where k is such that $\sigma_n = \sigma_k{}^\frown j$ for some j. Every continuous function F has a representative f as described above, and, in fact, it has infinitely many representatives. We define a measure μ^{**} on $\mathcal{F}$ induced by the standard probability measure on $\{0,1,2\}^{\mathbb{N}}$. Brodhead, Cenzer, and Remmel [5] defined an (Martin-Löf) random continuous function on $2^{\mathbb{N}}$ which has a representation in $\mathcal{F}$ which is Martin-Löf random. It is shown that any Martin-Löfrandom $r \in \{0,1,2\}^{\mathbb{N}}$ in fact represents a continuous function. The following results are proved in [2, 5]. Random Δ^0_2 continuous functions exist, but no computable function can be random and no random function can map a computable real to a computable real. The image of a random continuous function is always a perfect set and hence uncountable. For any $y \in 2^{\mathbb{N}}$, there exists a random continuous function F with y in the image of F. Thus the image of a random continuous function need not be a random closed set. The set of zeroes of a random continuous function is a random closed set (if nonempty).

There has been a considerable amount of work on studying reals whose complexity is "low" or trivial from the point of view of randomness. Chaitin defined a real x to be K-trivial if $K(x\lceil n) \leq K(1^n) + O(1)$. We recall that there are noncomputable c.e. sets which are K-trivial and that the K-trivial reals are downward closed under Turing reducibility. The latter is a highly non-trivial result of Nies [22] who also showed that the K-trivial reals form a Σ^0_3-definable ideal in the Turing degrees. In particular, this means that if α and β are K-trivial, then the join $\alpha \oplus \beta$ is also K-trivial.

The main goal of this paper is to study K-triviality for closed subsets of $2^{\mathbb{N}}$ and for continuous functions on $2^{\mathbb{N}}$. We define a closed set Q to be K-trivial if the code x_Q is K-trivial and we define a continuous function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ to be K-trivial if it has a representing function $f \in \mathcal{F}$ which is K-trivial. For example, in section two, we shall show that, for every K-trivial degree $\mathbf{d}$, there is a K-trivial Π^0_1 class of degree $\mathbf{d}$ which is the family of separating sets of a pair of computably inseparable c.e. sets. There also exist K-trivial Π^0_1 classes of positive measure with no computable member. Every K-trivial closed set contains a K-trivial member. There is no c.e. low set A such that the sets computed by A form a basis for the K-trivial classes. In section three, we shall show that for any K-trivial degree $\mathbf{d}$, there is a K-trivial continuous function on $2^{\mathbb{N}}$ with degree d. K-trivial functions map K-trivial reals to K-trivials. A closed set is K-trivial if and only if it is the set of zeroes of some K-trivial continuous function. In section four, we shall investigate the Medvedev degrees of difficulty of K-trivial Π^0_1 classes. There is no maximal or minimal K-trivial Medvedev degree. Finally, in section five, we give a new result on the members of random closed sets, namely that for $A, W \subseteq \omega$ if $W \leq_T A'$ then $W \equiv_T A'$ if and only if W can compute a path through every A-random closed set P such that $T_P \leq_T A'$.

2 K-trivial closed sets

Since every K-trivial real is Δ^0_2, we have that every K-trivial closed set is a strong Π^0_2 class. Note also that the canonical code of a Π^0_1 class has c.e. degree and that there are K-trivial reals with non-c.e. degree. Hence there are K-trivial closed sets which are not Π^0_1 classes.

Analogous to the existence of c.e. K-trivial reals, we will construct several examples of K-trivial Π^0_1 classes. Note that a Π^0_1 class P is said to be *decidable* if the canonical tree T_P is computable, which is if and only if the canonical code for P is computable. Thus we want to construct K-trivial Π^0_1 classes which are not decidable. The degree of a closed set Q is the degree of the tree T_Q and also the degree of the canonical code for T_Q. Since the K-trivial sets are closed under Turing reducibility, a closed set Q is K-trivial if and only if the tree T_Q is K-trivial. Thus we will also say that $Q = \emptyset$ is a K-trivial closed set.

We begin with those non-decidable Π^0_1 classes with the simplest structure, that is, countable classes with a unique limit path. Our first construction relies on the following notion. If $A = \{a_0 < a_1 < \cdots\}$ is an infinite set, then A is said to be *retraceable* if there is a partial computable function ϕ such that $\phi(a_{n+1}) = a_n$ for all n. The initial subsets of A are A together with the finite sets $\{a_0, \dots, a_{n-1}\}$ for each n. Dekker and Myhill [11] showed that every c.e. degree contains a retraceable Π^0_1 set A. Cenzer, Downey, Jockusch and Shore [8] showed that a Π^0_1 set A is retraceable if and only if the family $I(A)$ of initial subsets is a Π^0_1 class. Clearly $I(A)$ has unique limit element A.

Theorem 2.1. *For any noncomputable K-trivial c.e. degree $\mathbf{d}$, there exists a K-trivial Π^0_1 class P of degree $\mathbf{d}$ such that P has a unique, noncomputable limit element.*

Proof. Let A be a retraceable Π^0_1 set of degree $\mathbf{d}$. Then A is K-trivial and noncomputable and is the unique limit element of the Π^0_1 class $P = I(A)$ as shown above. It remains to show that the tree T_P has the same degree as A. Certainly $T_P \leq_T A$, since

$$\sigma \in T_P \iff (\forall i < |\sigma|)[\sigma(i) = 1 \rightarrow (i \in A \;\&\; (\forall j < i)(j \in A \rightarrow \sigma(j) = 1))].$$

On the other hand, $A \leq_T T_P$ since

$$a \in A \iff (\exists \sigma \in \{0,1\}^{a+1})(\sigma \in T_P \;\&\; \sigma(a) = 1).$$

□

We next construct a K-trivial class having only computable members.

Theorem 2.2. *For any K-trivial c.e. degree $\mathbf{d}$, there exists a K-trivial Π^0_1 class of degree $\mathbf{d}$ with unique limit path 0^ω and all elements computable.*

Proof. Let B be a co-c.e. set of degree $\mathbf{d}$ and let $Q = \{0^\omega\} \cup \{\{n\} : n \in B\}$. Clearly Q has all elements computable and unique limit element 0^ω. It is easy to check that $T_Q \equiv_T B$. □

Next we wish to obtain a Π^0_1 class with no computable members (a *special* Π^0_1 class) such that the code for the class is K-trivial. To do so we rely heavily on the fact that the K-trivials form an ideal in the Turing degrees (and in particular K-triviality is closed under Turing equivalence). It follows that the separating class for two disjoint K-trivial sets A, B will be K-trivial, as the set of its extendible nodes (and hence its code) is Turing-equivalent to $A \oplus B$. It remains to show there are recursively inseparable K-trivial sets. The following proof is due to Steve Simpson.

Theorem 2.3. *There is a K-trivial Π^0_1 class with no computable members.*

Proof. Let B be a noncomputable c.e. K-trivial set. Split B into disjoint noncomputable c.e. A_1, A_2 as in the Friedberg splitting theorem. Ohashi [24] observed that the proof of the Friedberg splitting theorem in fact gives that A_1, A_2 are recursively inseparable. By the downward closure of K-triviality, they are also K-trivial. Let S be their separating class. Then by the discussion above, S is a special K-trivial Π^0_1 class. $\square$

Note that the separating class constructed in Theorem 2.3 has measure zero. Next we construct K-trivial classes of arbitrarily large positive measure yet still containing no computable members. The proof makes use of the well-established *cost function* method from the area of algorithmic randomness, first used in Kučera-Terwijn [21] and later made explicit, e.g. in Downey-Hirschfeldt-Nies-Stephan [15].

Theorem 2.4. *There is a K-trivial Π^0_1 class (of arbitrarily large measure) with no computable paths (thus perfect).*

Proof. There is a well established framework for constructing K-trivial reals in the Cantor space 2^ω in terms of *cost functions*. A good presentation of this can be found in Nies [23]. It is clear that the same method applies to the space 3^ω. Let K be the prefix-free complexity and

$$cost(x,t) = \sum_{x<w\leq t} 2^{-K_t(w)}.$$

where $K_t(w)$ denotes the prefix complexity of w by stage s. It is well known that $\lim_x \sup_t cost(x,t) = 0$. In order to construct a K-trivial Π^0_1 class P it suffices to give a monotone approximation (P_t) to P (in the sense that $P_t \supseteq P_{t+1}$) such that if c_t is the code for P_t and x_s is the least number such that $c_{s-1}(x) \neq c_s(x)$ then

$$\sum_{s>0} cost(x_s, s) \leq 1. \tag{2}$$

Indeed in [23] it is shown that c is K-trivial iff it has a Δ^0_2 approximation (c_t) which satisfies (2). To make sure that there are no computable paths through P it suffices to satisfy the following requirements:

$$R_e : \Phi_e \text{ is total } \Rightarrow \ \Phi_e \notin P$$

where (Φ_e) is an effective enumeration of all Turing functionals with binary values. The strategy for R_e is to modify the code c at some stage so that the tree represented by c no longer extends some initial segment of Φ_e. This is done by switching a 2 in c to a 0 or 1 according to which has the desired effect. First note that each c_t will consist of all 2's except for a finite initial segment, so we will find a suitable digit to switch. Second note that when we change a position in c from 2 to something else (0 or 1), we can effectively adjust the tail of c (the digits after the modified digit) so that the code describes the tree that we get if we cut that branch from the branching node corresponding to the 2 above. This means that if we let R_e act on c in the way described above, we get an approximation to P which is co-c.e. (so P is a Π^0_1 class).

The last consideration is that R_e cannot change digit n at stage s unless $cost(n,s) < 2^{-(e+1)}$. This will make c K-trivial. Let $\mathbb{N}^{[e]}$ be the e-th column of $\mathbb{N}$, i.e. the set of numbers of the form $\langle e,t\rangle$ for some $t \in \mathbb{N}$ where $\langle .,.\rangle$ is a computable bijection from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$. The symbol $\restriction$ denotes restriction of the object that precedes it to the numbers $< x$. For example $\Phi_e \restriction x \downarrow$ means that Φ_e is defined at all arguments $< x$. All parameters in the construction are in formation and only have current values which correspond to the current stage.

Construction. At stage s look for the least $e < s$ such that R_e has not acted and there is a positive $x \in \mathbb{N}^{[e]}$ with the property that

- $\Phi_e \restriction x \downarrow$ and is on P_s
- $cost(k(x,s),s) < 2^{-(e+1)}$, where $k(x,s)$ is the position of node $\Phi_e \restriction (x-1)$ in the code c_s of P_s.

If there is no such e go to the next stage. Otherwise note that since R_e has not acted and $x \in \mathbb{N}^{[e]}$, no strategy has chopped any branch from node $\Phi_e \restriction (x-1)$ and so the latter is branching. Now switch $k(x,s)$ from 2 to $1 - \Phi_e(x-1)$ (so that $\Phi_e \restriction x \notin P$) and let larger positions describe the tree that we get by chopping that branch. Go to the next stage.

For the verification, the comments before the description of R_e explain why the approximation (c_t) defined in the construction corresponds to a co-c.e. approximation of P, so that P is a Π^0_1 class. Each R_e is satisfied by the standard cost-function argument: there is some x_0 such that for all $x > x_0$ and all s, $cost(x,s) < 2^{-(e+1)}$ (by the properties of $cost$). Finally c is K-trivial since the approximation (c_t) given in the construction satisfies (2) (that each R_e acts at most once and contributes cost at most $2^{-(e+1)}$). Finally note that by choosing the witnesses x sufficiently large we can make sure that P has measure arbitrarily close to 1. □

Theorem 2.5. *If P is a K-trivial Π^0_1 class then the leftmost path is a K-trivial real.*

Proof. The leftmost path is computable from the (code of the) Π^0_1 class P and since K-triviality is downward closed under Turing reductions it must be K-trivial. □

By Nies' top low$_2$ theorem (see [13]), there is a low$_2$ c.e. degree above all K-trivial degrees. By Theorem 2.5, this means that the sets computed by it form a basis for the K-trivial Π^0_1 classes (while no incomplete c.e. degree has this property with respect to all Π^0_1 classes). The following theorem shows that such a c.e. degree cannot be low. Note however that there are low PA degrees, i.e. low degrees such that the sets computed by them form a basis for all Π^0_1 classes. The corresponding problem for K-trivial reals—whether there is a low degree bounding all K-trivials—was recently answered positively by Kučera and Slaman [20].

Theorem 2.6. *If A is c.e. and low then there is a K-trivial Π^0_1 class which contains no A-computable paths. In other words, there is no c.e. low set A such that the sets computed by A form a basis for the K-trivial Π^0_1 classes.*

Proof. This is similar to the proof that for every c.e. low A there is a K-trivial B such that $B \not\leq_T A$ (in the same way that the proof of Theorem 2.4 is similar to the construction of a non-computable K-trivial set). If the reader is not familiar with that construction, (s)he might like to have a look at it [23]. We wish to follow the construction of Theorem 2.4 only now we need to satisfy the following more demanding requirements:

$$R_e : \Phi^A_e \text{ is total } \Rightarrow \Phi^A_e \notin P.$$

In general it is impossible to satisfy these requirements but if we know that A is low we can use the following trick (due to Robinson) to succeed. During the construction we will ask $\emptyset'$ a $\Sigma^0_1(A)$ question (for the sake of R_e). Note that since A is low, $\emptyset'$ can answer such questions. At each stage we will only have an approximation to $\emptyset'$ and so we will get a correct answer possibly after a finite number of false answers. Requirement R_e will use *witnesses* (in the sense of the proof of Theorem 2.4) from $\mathbb{N}^{[e]}$. We will ask the following:

> Is there a stage s and a witness x such that
>
> - $\Phi^A_e \upharpoonright x[s] \downarrow$ with correct A-use and $\Phi^A_e \upharpoonright x[s] \in P_s$
> - $cost(k(x,s),s) < 2^{-(n_e+e+3)}$
>
> where n_e is the number of times that some branch of P has been pruned (i.e. some digit of c has been changed) for the sake of R_e?

First notice that the above question refers to the partial computable sequences (P_s), $(n_e[s])$ which are defined during the very construction. By the recursion theorem we can ask such questions and approximate the right answers: given any partial computable sequence (P'_s) of Π^0_1 classes and uniformly partial computable sequences $(n'_e[s])$, we will effectively define a construction in which the questions refer to the given parameters. All of these constructions will define a sequence (P_s) of Π^0_1 classes which monotonically converges to a K-trivial Π^0_1 class P which however does not necessarily satisfy the other requirements; also each will define a uniformly partial computable sequence $(n_e[s])$. The (double) recursion theorem will give a construction in which the questions asked actually refer to (P_s) and $(n_e[s])$. Such a construction will succeed in satisfying all requirements. Let $g(e,s)$ be a computable function approximating the true answer to the questions above, when these are set to refer to the given parameters (P'_s), $(n'_e[s])$.

Construction. For stage s and each $e < s$ such that there is an *unused* $x \in \mathbb{N}^{[e]}$ satisfying $\Phi_e^A \upharpoonright x[s] \in P_s$ and $cost(k(x,s),s) < 2^{-(n_e[s]+e+3)}$ (where $n_e[s]$ is as above) do the following. Wait for a stage $t \geq s$ such that $g(e,t) = 1$ or the computation $\Phi_e^A \upharpoonright x[t]$ is different thant he computation $\Phi_e^A \upharpoonright x[s]$. In the first case switch $k(x,s)$ from 2 to $1 - \Phi_e(x-1)$ (so that $\Phi_e \upharpoonright x \notin P$) and let larger positions describe the tree that we get by chopping that branch (say that x has been *used*); proceed to stage $s+1$. In the latter case do nothing and test the next value of e. If the above has run over all $e < s$ and we are still at stage s, go to stage $s+1$.

For the verification, note that if x is unused at some stage, then currently all nodes of the xth level of P are branching. So each construction defines a (possibly finite) monotone sequence of clopen sets P_s (and so a Π^0_1 class P as a limit). Also, for every value of the input (P'_s), $(n'_e[s])$ the resulting class P is K-trivial as the condition (2) from the proof of Theorem 2.4 holds (at any stage at most one requirement acts and the cost of that action is small by construction). By the double recursion theorem there is a construction such that

$$n_e[s] = n'_e[s] \ \wedge \ P_s = P'_s$$

for all s, e; i.e. the input and output as (double) partial computable sequences are the same. This construction must be total (in the sense that it passes through all stages) since every search halts (for example if $\Phi_e^A \upharpoonright x[s] \in P_s$ with use u, $A[s] \upharpoonright u = A$ and $cost(k(x,s),s) < 2^{-(n_e[s]+e+3)}$ then $g(e)$ has to settle at 1 as it guesses correctly). Finally suppose that R_e is not satisfied. This means that the answer to the e-question is a negative one. So $g(e)$ would settle to 0 (since it approximates the correct answer to the e-question) and R_e would act finitely often. But then the cost requirement (in particular n_e) would remain constant and (by the properties of *cost*) for some large enough x, s the computation $\Phi_e^A[s] \upharpoonright x$ will be correct and $\Phi_e^A \upharpoonright x[s] \in P_s$, $cost(k(x,s),s) < 2^{-(n_e[s]+e+3)}$ which is a contradiction. □

3 K-trivial continuous functions

In [5], the notion of randomness was extended to continuous functions on $2^{\mathbb{N}}$. Thus it will be natural to consider K-trivial continuous functions. It was shown in [5] that a random continuous function maps any computable real to a random real. Note that Fouche [17] has earlier used a different approach to randomness for continuous functions connected with Brownian motion, first presented by Asarin and Prokovsky [1], and has shown that, under this approach, it is also true that for any random continuous function F, $F(x)$ is not computable for any computable input x. A general approach to randomness, which includes continuous functions, is developed by Gács [18]. The representation f for a continuous function F, as defined above in the introduction, may be viewed as a *labeled tree* where every node $\sigma \in \{0,1\}^*$ has a label from $\{0,1,2\}$ so that σ_n gets label $r(n)$.

Proposition 3.1. *For any K-trivial computable function F and any K-trivial real x, $F(x)$ is K-trivial.*

Proof. Let f be the K-trivial function which codes F. Then $y = F(x)$ may be computed uniformly from f and x as follows. For each m, search for a large enough k such that $f(x\lceil k)$ has at least m values in $\{0,1\}$ and letting $y(m)$ be the mth such value. Then it follows from the closure under join of K-trivial degrees that $f \oplus x$ is also K-trivial and then by the downward closure that F is K-trivial. □

It was shown in [5] that the set of zeroes of a random continuous function is either empty or random.

Proposition 3.2. *For any K-trivial continuous function F, the set of zeroes of F is a K-trivial closed set.*

Proof. Let F be K-trivial and let $Q = \{x : F(x) = 0\}$. The tree T_Q may be defined from the K-trivial representative r for F as follows. T_Q contains a node σ if and only if for all $\sigma_n = \tau \sqsubset \sigma$, $r(n) \neq 1$. Clearly $T_Q \leq_T r$, so that Q is K-trivial by the downward closure under Turing reducibility. □

This is actually a characterization of the K-trivial closed sets, by the following.

Proposition 3.3. *For any K-trivial closed set Q, there is a K-trivial continuous function F such that Q is the set of zeroes of F.*

Proof. Given the K-trivial tree T_Q, we simply define the desired K-trivial continuous function F by the labelled tree which puts a "0" on all nodes of T_Q and puts a "1" on every other node. Then F is computable from T_Q and therefore is K-trivial. □

We consider a continuous functions $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ always in terms of one its representing functions $f : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, or, equivalently, in terms of the code of one of its representing functions. Note that by slowing the convergence of the function on finite strings, we may code information into the code of the function. Hence the codes of a given function on Cantor space are always closed upwards in the Turing degrees, so the K-degree of a function should be the K-degree of the *canonical* representation, and corresponding canonical code, that which converges as rapidly as possible. This may be defined as follows. Each $\sigma \in \{0,1\}^*$ is mapped to τ, where τ is the longest string of length $\leq |\sigma|$ such that, for all $x \in I(\sigma)$, $F(x) \in I(\tau)$. It is clear that the canonical representation of a function F may be computed from *any* representation. That is, given a representation f and input σ, search for n large enough so that $|f(\rho)| \geq |\sigma|$ for all extensions ρ of σ of length n and let $g(\sigma)$ be the common part of those $\{f(\rho) : \rho \in \{0,1\}^n : \sigma \sqsubseteq \rho\}$. It follows from the downward closure of K-triviality that F is K-trivial if and only if the canonical code is K-trivial.

Theorem 3.4. *For any K-trivial degree* $\mathbf{d}$*, there is a continuous function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with canonical code of degree* $\mathbf{d}$*. Moreover, if* $\mathbf{d}$ *is c.e., F may be chosen to have left-c.e. canonical code.*

Proof. Let $A = \{a_1, a_2, \ldots\}$ be a set of degree $\mathbf{d}$. We define F monotonically increasing such that $F(0^\omega) = 0^\omega$ and $F(1^\omega) = \chi_A$, the characteristic function of A. We will construct the representing function $f : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$. To begin, let $f(0) = 0^{(a_1+1)}$ and $f(1) = 0^{a_1}1$. Now suppose we have defined $f(\sigma) = \tau$ for $|\sigma| = n-1$, and that $a_n - a_{n-1} = m$. Then let $f(\sigma 0) = \tau 0^m$ and $f(\sigma 1) = \tau 0^{m-1}1$. It is clear that $f \equiv_T A$, so f is of degree $\mathbf{d}$. Furthermore, if $\mathbf{d}$ is a c.e. degree and A is chosen c.e., the code given by f will be left-c.e., as shown by an analysis of the construction.

The code for f may be thought of as composed of blocks of length 2^n for $n \geq 1$, in order of increasing size, corresponding to different levels of the tree. At level n, if $n-1 \notin A$, the block will be all zeros. If $n-1 \in A$ and $card(A \cap \{0,1,\ldots,n-1\}) = k$, the block will consist of 2^k subblocks of 2^{n-k} bits each, beginning with a subblock of all zeros and alternating to end with a subblock of all 1s. Thus the structure of the n^{th} block is determined entirely by whether $n-1$ is in A, and if so, how many values $< n-1$ are also in A.

Given an enumeration of A as A_s, $s \in \omega$, we may define an approximation to the function F with corresponding canonical code C_s. We show that as s increases, a bit of C_s holding a one may only change to zero if a preceding bit changes from zero to one; this shows that C_s is an increasing approximation. As the enumeration A_s is computable by assumption, the canonical code of F is then left-c.e. Without loss of generality we consider a single level of the tree, n, and a single stage, s. If the corresponding block of C_{s-1} is all zeros, this level causes no trouble at stage s: either it remains all zeros or half of its zeros change to ones. If the n^{th} block of C_{s-1} is half zeros and half ones, then enumeration into A at stage s may cause the subblocks to multiply and rearrange. However, this only occurs when some $k < n-1$ enters A_s, causing the corresponding earlier level to change from all zeros to half zeros and half ones. □

4 Medvedev degrees of K-trivial classes

The degrees of difficulty of K-trivial closed sets should be of interest. Simpson [25], Cenzer and Hinman [9] and others have developed the subject of the Medvedev (or strong) degrees of Π^0_1 classes. Here $P \leq_M Q$ means that there is a computable function mapping Q into P. The Medvedev degrees form a lattice where the meet operation is the disjoint union

$$P \wedge_M Q = P \oplus Q = \{0^\frown x : x \in P\} \cup \{1^\frown x : x \in Q\}$$

and the join is the product,

$$P \vee_M Q = P \otimes Q = \{x \oplus y \mid x \in P \text{ and } y \in Q\}.$$

There is a least degree 0_M consisting of the classes with a computable member and a highest degree 1_M which can be viewed as a universal Π^0_1 class (for example, the set of complete consistent extensions of Peano Arithmetic). A related structure are the *Muchnik (or weak) degrees*, where P is weakly reducible to Q ($P \leq_w Q$) if for every $\beta \in Q$ there exists $\alpha \in P$ such that $\alpha \leq_T \beta$.

One general problem is where the K-trivial Π^0_1 classes fit into the Medvedev (or Muchnik) degrees of the Π^0_1 classes. We have only a few results so far. Since the K-trivial reals form an ideal in the Turing degrees, it follows that the family of Π^0_1 classes which contain a K-trivial real form an ideal in the lattice of Medvedev degrees (and also in the lattice of Muchnik degrees). The following proposition says that the K-trivial Π^0_1 classes are closed under the meet and the join operation in the Medvedev degrees.

Proposition 4.1. *The K-trivial Π^0_1 classes are closed under disjoint unions and under products.*

Proof. The degree of the code of the disjoint union of two Π^0_1 classes is the join of the degrees of the codes of these Π^0_1 classes. The same holds for products and since K-triviality is invariant in the Turing degrees and closed under join (in the Turing degrees) the proposition follows. □

Note however that K-triviality (for Π^0_1 classes) is not closed under Medvedev equivalence. For example the least Medvedev degree contains Π^0_1 classes with computable leftmost path but with a canonical code which computes the halting problem. Hence we will call a Medvedev degree K-trivial if it contains a K-trivial class. Since there is no c.e. complete K-trivial real and any Medvedev complete Π^0_1 class is also c.e. complete, it follows that no K-trivial Π^0_1 class is Medvedev complete. A relevant question is whether there a greatest Medvedev degree among the K-trivials, or even a maximal one.

Theorem 4.2. *There is no maximal K-trivial Medvedev or Muchnik degree.*

Proof. Given any K-trivial Π^0_1 class Q it suffices to construct a K-trivial Π^0_1 class P which is not weakly reducible to Q. Indeed, in that case $P \otimes Q$ would be K-trivial strongly above Q and not weakly below Q. We argue as follows. By the Low Basis Theorem, Q contains a member α of low Turing degree. Now by Theorem 2.6, there is a K-trivial Π^0_1 class P with no path computed by α. This means that P is not weakly reducible to Q. □

The above proof also shows that there is no Π^0_1 class P which has low canonical code and is weakly above all K-trivial Π^0_1 classes.

Next we show that there is also no minimal K-trivial Medvedev or Muchnik degree. The following combines Lemma 2 and Lemma 5 from [4]. It is used there to show that any Π^0_1 class $P \neq_M 0_M$ may be split into subclasses P_0 and P_1, each properly below P and joining to P.

Lemma 4.3 (Binns). *Let P be any Π^0_1 class with no computable element and let A be any c. e. set. Then there exist Turing incomparable c.e. sets A^0 and A^1 such that (i) $A^0 \cup A^1 = A$ and $A^0 \cap A^1 = \emptyset$ and (ii) for each $i \in \{0,1\}$ and $x \in P$, $x \not\leq_T A^i$. Furthermore, if Q is any Π^0_1 class such that $Q <_M P$, then we can choose A^0 and A^1 such that $P \not\leq_M (\{A^i\} \otimes Q)$ for $i = 0, 1$.*

We need another fact about separating classes.

Lemma 4.4. *For any sets A, B, if A^0, A^1 is a splitting of A, then $S(A^0, B) \otimes S(A^1, B) \equiv_M S(A, B)$.*

Proof. For the one direction, note that $S(A,B) \subset S(A^i, B)$, so that a map from $S(A,B)$ into $S(A^0,B) \otimes S(A^1,B)$ may be given by $F(X) = X \oplus X$. For the other direction, given $X_0 \oplus X_1$ such that $X_i \in S(A^i, B)$, let $G(X_0 \oplus X_1) = X_0 \cup X_1 \in S(A,B)$. □

Theorem 4.5. *For any K-trivial Π^0_1 class P with no computable member, there exists a K-trivial Π^0_1 class Q such that $0 <_M Q <_M P$ and similarly for the Muchnik degrees.*

Proof. Given the K-trivial Π^0_1 class P, let A and B be disjoint computably inseparable K-trivial c. e. sets so that $S(A,B)$ is a K-trivial Π^0_1 class as in Theorem 2.3. Now apply Lemma 4.3 to obtain A^0 and A^1 and observe that each $A^i \leq_T A$ and is therefore also K-trivial. Now let $P^i = P \oplus S(A^i, B)$ for $i = 0, 1$; each $S(A^i, B)$ is K-trivial as in Theorem 2.3 and thus each P^i is K-trivial by Proposition 4.1. Certainly each $P^i \leq_M P$ since $S(A^i, B) \leq_M S(A,B)$, but $P \not\leq P^i$, since $P^i \not\leq P$ since $0^\frown A^i \in P^i$ and $x \not\leq_T A^i$ for each i. Thus $P^i <_M P$. On the other hand, it follows from distributivity that $P^0 \vee P^1 \equiv_M P \vee S(A,B) \neq_M 0_M$, so that at least one of P^0 and P^1 has no computable member. Let Q be P^i such that $P^i \neq_M 0_M$. □

We can also obtain a splitting result for K-trivial separating classes.

Theorem 4.6. *For any K-trivial Π^0_1 separating classes $Q <_M P$, there exist K-trivial Π^0_1 classes P^0 and P^1 such that $P <_M P^0, P^1 <_M Q$ and $P^0 \vee P^1 \equiv_M P$.*

Proof. Let $Q = S(A,B)$ where A and B are K-trivial, let A^0 and A^1 be given by Lemma 4.3 and let $P^i = (S(A^i, B) \vee Q)$. Since $Q <_M P$, it follows from the proof of Theorem 4.5 that each $P^i <_M P$. It follows from the furthermore clause of Lemma 4.3 that $P \not\leq_M S^i \vee Q$, so that each $P^i <_M P$. Applying distributivity and Lemma 4.4, we have $P^0 \vee P^1 \equiv_M P$. Now suppose by way of contradiction that $P^i \leq_M Q$. Then $P^i \leq P^{1-i}$, so that $P \equiv_M P^0 \vee P^1 = P^{1-i} <_M P$, a contradiction. Thus $Q <_M P^i$ for each i. □

To get a splitting for an arbitrary P, we would need a K-trivial separating class above P. By Theorem 4.2, there is no single such class. The question of whether every K-trivial Π^0_1 class is $\leq_M S(A,B)$ for *some* K-trivial separating class may be of interest.

5 Random closed sets

We conclude with some results about the degrees of the members of a random closed set.

Theorem 5.1. *If P is a random closed set then there exists $\beta \in P$ such that $\beta <_T P$, where P is identified with its canonical code.*

Proof. Let T_P be the tree without dead ends which represents P and consider the least branching node σ of T_P. The restriction T_* of T_P below σ has the same degree as T_P and is random. Now let $T_* = T_0 \boxplus T_1 = \{0^\frown\sigma : \sigma \in T_0\} \cup \{1^\frown\sigma : \sigma \in T_1\} \cup \{\emptyset\}$, as defined in [3] where T_i is the restriction of T_* below $\sigma * i$. By a result in [3] we have that T_i are random relative to each other. In particular, $T_i \not\leq_T T_{1-i}$ and so, $T \not\leq_T T_i$ for $i = 0, 1$. Then if β is the leftmost path of T_0 we have $\beta \leq_T T_0 \leq_T T$ and $T \not\leq_T \beta$, so that $\beta <_T T$. ☐

Theorem 5.1 shows in particular that every Δ^0_2 random closed set has an incomplete Δ^0_2 path. Nevertheless the following theorem shows that for any incomplete Δ^0_2 degree, there is a random closed set with no paths computable from that degree.

Theorem 5.2. *Let A be an oracle and let $W \leq_T A'$. Then $W \equiv_T A'$ iff W can compute a path through every A-random closed set P with $T_P \leq_T A'$.*

Proof. We show the theorem for $A = \emptyset$ as a relativisation is straightforward. First of all $\emptyset'$ can compute a path through every Δ^0_2 random closed set, as observed in [3]. For the other direction it is enough to show that if $W <_T \emptyset'$ then there is a Δ^0_2 random closed set P such that $\beta \not\leq_T W$ for all $\beta \in P$. We construct the canonical code of P recursively in oracle $\emptyset'$. By [19] there is a Π^0_1 class L in the space 3^ω (the space of canonical codes) which contains only random sequences, and a recursive function f such that $\mu([\sigma] \cap P) > 2^{-f(|\sigma|)}$ or $[\sigma] \cap P = \emptyset$, for all $\sigma \in 3^\omega$ (μ is the standard measure on 3^ω). Recall from [3] that $\mu^*(\{Q \in \mathcal{C} \mid Q \cap [\tau] \neq \emptyset\}) = \left(\frac{2}{3}\right)^{|\tau|}$ and let $g(n)$ be the least t such that $\left(\frac{2}{3}\right)^t < 2^{-f(n)}$. Also let $h(n)$ be the least t such that $\emptyset' \restriction n = \emptyset'[t] \restriction n$, Y_n the set of $\sigma \in 3^{<\omega}$ of length n such that $[\sigma] \cap L \neq \emptyset$ and p_s the largest p such that $\Phi^W_e[s] \restriction p \downarrow$. We construct a random $x_P \in 3^\omega = \cup_s \sigma_s$ such that $|\sigma_s| \leq s$, $\sigma_s \subseteq \sigma_{s+1}$, $\sigma_0 = \emptyset$ for all s, and

$$R_e : \Phi^W_e \text{ is total } \Rightarrow \Phi^W_e \notin P$$

for all $e \in \mathbb{N}$, where Φ_e is an effective enumeration of all Turing functionals with binary values. We say that R_e *requires attention* at stage $s+1$ if

$$\{Q \in \mathcal{C} \mid [\Phi^W_e \restriction p_s] \cap Q \neq \emptyset \text{ and } \sigma_s \subset x_Q\} \neq \emptyset \qquad (3)$$

(i.e. there are extensions of σ_s which do not satisfy R_e) and either $\Phi^W_e[h(s)] \restriction g(s) \downarrow$ or R_e received attention at s. We order the strings in $3^{<\omega}$ as usual, first by length and then lexicographically.

Construction At stage $s+1$ consider the least $e < s$ such that R_e requires attention at $s+1$. If there is an extension ρ of σ_s which belongs to Y_s and

$$\{Q \in \mathcal{C} \mid [\Phi_e^W \upharpoonright g(s)] \cap Q \neq \emptyset \text{ and } \rho \subset x_Q\} = \emptyset \tag{4}$$

then let $\sigma_{s+1} = \sigma_s {}^\frown \rho(|\sigma_s|)$ for the least such ρ. Otherwise, or even if there is no $e < s$ such that R_e requires attention, let $\sigma_{s+1} = \sigma_s$.

Verification By construction we have that $[\sigma_s] \cap L \neq \emptyset$ for every s and the sequence (σ_s) is recursive in $\emptyset'$. So there is some random closed set P with $x_P \in \cap_s[\sigma_s]$. First we show that if R_e stops requiring attention after some stage s_0 then it is satisfied by all closed sets P with $x_P \in \cap_s[\sigma_s]$. Indeed, this is the case if Φ_e^W is not total, and if on the other hand it is total, there would be some $s_1 > s_0$ such that $\Phi_e^W[h(s_1)] \upharpoonright g(s) \downarrow$ (otherwise W would compute a function which dominates h, thus computing $\emptyset'$, a contradiction). By hypothesis R_e does not require attention at $s+1$ and so we must have that (3) does not hold which means that all extensions of σ_s, hence x_P, satisfy R_e.

Now we can show by induction that for each e the requirement R_e is satisfied by all closed sets P with $x_P \in \cap_s[\sigma_s]$. We note that it follows that $\lim_s |\sigma_s| = \infty$ (and so there is only one such x_P), since otherwise $lim_s \sigma_s = \sigma$ for some finite string σ and then the closed set P with code $\sigma {}^\frown 0^\omega$ clearly possesses a computable element x and does not satisfy the requirement R_e when $\Phi_e^W = x$. By the previous discussion it suffices to show that R_e requires attention finitely often, for each e. Suppose, for a contradiction, that there is a least e such that R_e requires attention infinitely often and let s_1 be the least stage after which no R_i, $i < e$ requires attention. Then (3) holds for all s. If s_2 is the least stage $\geq s_1$ at which R_e requires attention, by the choice of s_1 the requirement R_e did not receive attention at $s_2 - 1$ and so we must have $\Phi_e^W[h(s_2)] \upharpoonright g(s) \downarrow$. Then R_e requires attention at all stages $s \geq s_2$. Since $\left(\frac{2}{3}\right)^{g(s_2)} < 2^{-f(s_2)}$ and $|\sigma_{s_2}| \leq s_2$ there is some $\rho \supset \sigma_{s_2}$ such that $[\rho] \cap L \neq \emptyset$ and (4) holds with $s = s_2$. At some stage $s_3 \geq s_2$ we will have $\rho \in Y_{s_3}$ and $\sigma_{s_2} = \sigma_{s_3}$ so the contruction will set $\sigma_{s_4} = \rho$ at some stage $s_4 \geq s_3$ and for all $s \geq s_4$ (3) will not hold. Hence R_e will not require attention after s_4, which is a contradiction. This concludes the induction and the proof of the theorem. $\square$

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