Π^0_1 Classes and pseudojump operators

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Abstract

For a pseudojump V^X and a Π^0_1 class P, we consider properties of the set $\{V^X : X \in P\}$. We show that if P is Medvedev complete or if P has positive measure, and $\emptyset' \leq_T C$, then there exists $X \in P$ with $V^X \equiv_T C$. We examine the consequences when V^X is Turing incomparable with V^Y for $X \neq Y$ in P and when $W_e^X = W_e^Y$ for all $X, Y \in P$. Finally, we give a characterization of the jump in terms of Π^0_1 classes.

Keywords: Computability, Π^0_1 Classes

1 Introduction

The study of pseudojumps is a natural extension of the study of c.e. sets and degrees, which are fundamental in computability theory. These operators have been of particular interest in computability theory since the seminal papers [8] and [9] by Jockusch and Shore. Although it is not usual in the literature, it will be useful for us to make a distinction between pseudojumps and the more restricted class of CEA operators. If ϕ_e^X is the eth partial computable functional with oracle X, then $W_e^X = \{n : \phi_e^X(n) \downarrow\}$ is an c.e. operator. If $X \leq_T W_e^X$, then W_e^X is relatively computably enumerable in and above X. If this holds for every $X \in 2^{\mathbb{N}}$, then W_e^X is said to be a *pseudojump*. In particular, the jump operator $J(X) = X' = \{e : \phi_e^X(e) \downarrow\}$ is a pseudojump. Since it is a noncomputable question to decide in general whether or not a given c.e. operator is a pseudojump, the notion of CEA operator is sometimes more convenient. For any index $e \in \omega$, the *eth CEA operator*, J_e, maps X to $J_e(X) = X \oplus W_e^X$. Thus every CEA operator is a pseudojump, and every pseudojump operator has a Turing-equivalent CEA operator. We will often denote an arbitrary pseudojump by V. Friedberg in [4] constructed a noncomputable c.e. set A such that $A' \equiv_T \emptyset'$. The fundamental theorem for CEA operators, from [8], states that for any index e, there exists a noncomputable c.e. set A such that $W_e^X \equiv_T \emptyset'$, which generalizes the result of Friedberg. On the other hand, if V is obtained by relativizing the construction of a noncomputable low set, then $(V^A)' = A'$, so that if $V^A \equiv_T \emptyset'$, then $A' = \emptyset''$. In each of these examples, $X <_T V^X$ for all X. We will say that a pseudojump V is *strongly nontrivial* if $X <_T V^X$ for all X. V is *weakly nontrivial* if $X <_T V^X$ for all c.e. X. In the recent paper [2], it was shown that for any weakly nontrivial pseudojump V, there exist Turing incomparable c.e. sets A and B such that $V^A \equiv_T V^B \equiv_T \emptyset'$.

Another important area of study in computability theory is that of effectively closed sets of reals, the so-called Π^0_1 classes, which play an important role in many areas of computable mathematics. Characterizing the possible degrees of members of Π^0_1 classes is of great interest here. For example, every Π^0_1 class $Q \subseteq 2^{\mathrm{N}}$ has a member of c.e. degree, but there exist Π^0_1 classes with no computable member. A survey of results on Π^0_1 classes may be found in [1]. For each partial computable function $\phi_e : {}^{<\omega}2 \rightarrow 2$, let $T_e = \left\{ \tau : \forall \sigma \subset \tau \phi_{e,|\tau|}(\sigma) \neq 1 \right\}$; in this case $P_e = \left\{ x : x \in [T_e] \right\}$ is a uniform enumeration of all Π^0_1 classes. The topology here is given by a basis of clopen sets of the form $I(\sigma) = \{X : \sigma \subset X\}$ for $\sigma \in 2^{<\omega}$, where $\sigma \subset X$ means that $\sigma(i) = X(i)$ for all $i < |\sigma|$.

In this paper, we consider the interaction between pseudojumps and Π^0_1 classses, in particular how pseudojumps act on Π^0_1 classes. Recent work of Simpson [13] on the Medvedev degrees of Π^0_1 classes has characterized the complete degrees in several ways. Below, we show that if V is a CEA operator, P is a Medvedev complete Π^0_1 class, and C is any set in which $K = \emptyset'$ is computable, then there exists $X \in P$ with $V^X \equiv_T X \oplus \emptyset' \equiv_T C$. A related result was proved by Jockusch and Soare in [7], where they showed that the jump operator has this property for any Π^0_1 class having no recursive member.

For any Π^0_1 class P of positive measure, we again show that for any CEA operator V^X and any $C \geq_T K$, there exists $X \in P$ with $V^X \equiv_T X \oplus \emptyset' \equiv_T C$. Downey and Miller [3] recently obtained a stronger result for Σ^0_2 sets C and for $V^X = X'$ by making $X \leq_T K$. Note that a class of positive measure is not Medvedev complete and that a class of positive measure may or may not contain a recursive member.

We also consider for a Π^0_1 class Q, properties of the set $\{V^X : X \in Q\}$. by examining the consequences of having $W^X_e = W^Y_e$ for all $X \in Q$ and of having W^X_e Turing incomparable with W^Y_e for all $X \neq Y$ in Q. Finally, we give a new characterization of the jump in terms of Π^0_1 classes and discuss a method for defining pseudojumps in terms of Π^0_1 classes.

It is easy to find a nonempty Π^0_1 class P and a CEA operator V such that $V^X \not\equiv_T \emptyset'$ for any $X \in P$. For example, if P contains only computable elements and V^X is low^X, then $(V^X)' \equiv \emptyset'$ for all $X \in P$. On the other hand, if P is complicated enough, then it should have a member with $V^X \equiv_T \emptyset'$.

For Π^0_1 classes with no computable members, we still might not have a c.e. member or even a member of c.e. degree with $V^X \equiv_T \emptyset'$. We can find examples of such *special* Π^0_1 classes with no members X of c.e. degree such that $V^X \equiv_T \emptyset'$. Jockusch [5] constructed a Π^0_1 class P with no c.e. members at all. Jockusch and Soare [6] constructed a Π^0_1 class Q such that for any c.e. degree $\mathbf{b}$ and any $X \in P$, if $\deg(X) \leq \mathbf{b}$, then $\mathbf{b} = \emptyset'$. Thus if X has c.e. degree and $X \in Q$, then $X \equiv_T \emptyset'$, so that if $V^X \leq_T \emptyset'$, then $V^X \equiv_T X$, so that V fails to be strongly non-trivial. Recall that the Low Basis Theorem of Jockusch and Soare [7] shows that any nonempty Π^0_1 class $P \subseteq 2^{\mathrm{N}}$ must contain a member of low degree. More generally, In fact, it is also shown in [7] that if the Π^0_1 class P has no computable elements, then for any C above $\emptyset'$, P contains an element X such that $X' \equiv_T X \oplus \emptyset' \equiv_T C$.

2 Medvedev complete sets and pseudojumps

Since $V^X \leq_T X'$ for any set X and any pseudojump V, the following is an immediate corollary of the low basis theorem. We sketch a proof in preparation for the main theorem. Let K denote the halting problem $\{e : \phi_e(e) \downarrow\}$.

Proposition 1. *For any pseudojump V and any nonempty Π^0_1 class P, there exists $X \in P$ with $V^X \leq_T K$.*

Proof. This is an simple modification of the Low Basis Theorem of [7]. Let $P = [T]$ and fix e such that $V^X = W^X_e = \{n : \phi^X_e(n) \downarrow\}$. For each n, define the computable tree

$$U_n = \{\sigma \in \{0,1\}^* : \phi^\sigma_e(n) \uparrow\}.$$

Then $[U_n] = \{X : \phi^X_e(n) \uparrow\}$. Now define a sequence of Π^0_1 trees $\{S_n : n < \omega\}$ as follows. Let $S_0 = T$ and for each n, define

$$S_{n+1} = \begin{cases} S_n \cap U_n, & \text{if } S_n \cap U_n \text{ is infinite,} \\ S_n, & \text{otherwise.} \end{cases}$$

Now let $S = \cap_n S_n$ and $Q = [S] = \cap_n [S_n]$. By assumption, P is nonempty so that T is infinite and it follows from the construction, by induction, that each S_n is infinite. Thus Q is nonempty.

The construction is computable in K and therefore $\{n : S_n \cap U_n \text{ is infinite}\}$ is computable in K. Now for $X \in [S_{n+1}]$, it is clear that if $S_n \cap U_n$ is infinite, then $n \notin V^X$. On the other hand, if $S_n \cap U_n$ is finite, then $[S_n] \cap [U_n] = \emptyset$, so that for $X \in [S_n]$, $n \in V^X$. This gives a computation of V^X using K. Note that for any $X, Y \in Q$, we have $V^X = V^Y$. $\square$

For any computable set R, $\{R\}$ is a Π^0_1 class; so, unless V^Y has the same degree as Y' on all computable Y, $V^X <_T K$ for the X above. Hence there is no hope of finding a set in a given Π^0_1 class completing an arbitrary pseudojump operator without some condition guaranteeing the "richness" of the class.

We now turn to the main result. Let $\mathcal{B}$ be the computable Boolean algebra of clopen sets in $\{0,1\}^{\mathrm{N}}$. A clopen set is simply a finite union of intervals. A Π^0_1 class P is said to be *productive* if there is a computable *splitting* function $g : \mathrm{N} \to \mathcal{B}$ such that, for any e, if $P_e \cap P$ is nonempty, then both $P_e \cap P \cap g(e)$ and

$P_e \cap P - g(e)$ are nonempty. Simpson [14] showed that a Π^0_1 class is productive if and only if it is Medvedev complete. The Medvedev complete classes are the most *difficult* in the sense that if Q is Medvedev complete and P is any Π^0_1 class, then there exists a computable map Φ mapping Q into (in fact, onto) P. It is not hard to directly construct a nonempty Π^0_1 class that is Medvedev complete by uniformly coding a standard enumeration of the Π^0_1 classes as the infinite paths in a single recursive tree, as in Simpson [14], Lemma 3.3. There are many other examples, for instance, classes consisting of completions of Peano Arithmetic are Medvedev complete.

By combining the idea for the proposition above with a method for coding information into V, we can prove an analogue of Friedberg's jump theorem for Medvedev complete classes.

Theorem 1. *Let V be a pseudojump, $K \leq_T C$, and P be a Medvedev complete Π^0_1 class. Then there exist infinitely many $X \in P$ with $V^X \equiv_T X \oplus K \equiv_T C$.*

Proof. Let $P = P_c = [T]$ be Medvedev complete and let g be a splitting function for P. Let $\mathcal{Q}$ be the usual basis of clopen sets for $2^{\mathbb{N}}$, $\left\{ I(\sigma) \,:\, \sigma \in {}^{<\omega}2 \right\}$. We now modify the proof of Proposition 1 above so that the set C will be coded into V^X via a function $f : \mathrm{N} \to \mathcal{Q}$, computable in V^X, such that

$$X \in f(n) \text{ if and only if } n \in C.$$

Fix e such that $V^X = W^X_e$ and let $U_a = \left\{ \sigma \in \{0,1\}^* \,:\, \phi^{\sigma}_{e,|\sigma|}(a)\uparrow \right\}$, just as in Proposition 1 as above. Now define the sequences $\{R_n : n < \omega\}$ and $\{Q_n : n < \omega\}$ of Π^0_1 classes as follows. Let $Q_0 = P = P_c$ and let

$$R_n = \begin{cases} Q_n \cap [U_n], & \text{if } Q_n \cap [U_n] \text{ is nonempty,} \\ Q_n, & \text{otherwise.} \end{cases}$$

Let $R_n = P_{r(n)}$. By the construction, R_n is a nonempty subset of P, so that $R_n \cap [g(r(n))]$ and $R_n \cap [\overline{g(r(n))}]$ are both nonempty subsets of P. Then define

$$Q_{n+1} = \begin{cases} R_n \cap [g(r(n))], & \text{if } n \in C, \\ R_n \cap \overline{[(g(r(n))]}, & \text{otherwise.} \end{cases}$$

As before, let $Q = \cap_n Q_n$. Clearly, each R_n and Q_n are nonempty closed sets of reals and hence Q is nonempty. Let $X \in Q$. For any Π^0_1 class P we write T_P for the computable tree such that $P = [T_P]$.

First, we claim $V^X \leq_T C$. Note that $r(0) = c$ if and only if $0 \in V^X$, which happens if and only if the computable tree $T_{Q_0} \cap U_0$ is finite. This can be determined by $K \leq_T C$. Hence C can determine $V^X(0)$ and the index $r(0)$ of R_0. Now, if $0 \in C$, $Q_1 = R(0) \cap [g(r(0)]$, and, if $0 \notin C$, $Q_1 = R(0) \cap \overline{g(r(0))}]$. Since g is a computable function, C can therefore compute an index $q(1)$ for Q_1. Using this as a model, it follows by a routine induction that functions r and q giving the indices of R_n and Q_n respectively for all n are computable in C. Since $n \in V^X$ if and only if $T_{Q_n} \cap U_n$ is finite, and this fact is computable in $K \leq_T C$, $V^X \leq_T C$.

Next, we claim $V^X \leq_T X \oplus K$. The only place where we use C rather than K in the previous argument is to determine the index of $T_{Q_{n+1}}$. But if $X \in [g(r(n))]$, $Q_{n+1} = R_n \cap [g(r(n))]$; and if $X \notin [g(r(n))]$, $Q_{n+1} = R_n \cap \overline{[g(r(n))]}$. Since g is a computable function, this is computable in X, so that $q(n+1)$ is computable in $X \oplus K$.

Notice that as $X \leq_T V^X$, and K, $V^X \leq_T C$, we have $V^X \leq_T X \oplus K \leq_T C$. By definition of Q_{n+1}, $n \in C$ if and only if $X \in [g(r(n))]$, so to show that $C \leq_T V^X$, we need only show that the index function r is computable in V^X. But this follows again from a routine induction: knowing $q(n)$, V^X can determine the index $r(n)$, since that depends on whether or not $n \in V^X$, and, knowing the index $r(n)$, V^X can determine the index $q(n+1)$, since that depends on whether or not $V^X \in [g(r(n))]$ and g is a computable function.

To obtain infinitely many X with $V^X \equiv_T X \oplus K \equiv_T C$, note that for any σ such that $P \cap I(\sigma) \neq \emptyset$, $P \cap [\sigma]$ is also Medvedev complete. This is because the splitting function for P is easily adapted to a splitting function for $P \cap [\sigma]$. This means that for every σ such that $P \cap [\sigma] \neq \emptyset$, there exists $X \in [\sigma]$ as required. Thus there are infinitely many such $X \in P$. $\square$

Many properties indicating the richness of Medvedev complete classes follow directly from the theorem. As in the case of the similar result 2.1 in [8], Theorem 1 can be used to show that virtually every phenomenon in the c.e. degrees occurs inside of every Medvedev complete class. For example, suppose V is an operator that constructs a noncomputable low c.e set, and $C = K$. Then the set X given by the theorem must have high degree, since K is low over it. Since such a V is uniformly nontrivial, $X <_T K$. Hence every Medvedev complete class must contain an incomplete high degree.

Using Theorem 1 makes it possible to show many other facts about members of Medvedev complete classes. For instance, it follows immediately from Theorem 1 that every Medvedev complete class must have a nonarithmetic member, since we can take C to be the hyperjump. More generally, taking V to be the trivial operator in Theorem 1 actually shows that any Medvedev complete Π^0_1 class contains sets of every degree above $\emptyset'$, a fact due to Simpson, in [14], where he uses his characterization of Medvedev complete Π^0_1 classes by means of productive funcions to show that every Medvedev complete Π^0_1 class is effectively homeomorphic to the class of completions of Peano Arithmetic. The fact then follows from the fact that the class of completions of Peano Arithmetic is closed upward under Turing reducibility, a result due to Solovay.

The set X constructed in Theorem 1 is Δ^0_2 if and only if $C \equiv_T K$. Since any Π^0_1 class must contain a set of c.e. degree, we would like to strengthen Theorem 1 as in [8] complete pseudojump operators with an X of c.e. degree in a given

Medvedev complete set. However, Jockusch and Soare showed in [7] that the only c.e. degree that contains a completion of Peano Arithmetic is $\mathbf{0}'$, so the result of Simpson shows that this is impossible.

3 Classes with positive measure

In this section, we consider pseudojump inversion for Π^0_1 classes of positive measure. Downey and Miller recently proved that for any Π^0_1 class P of positive measure and any Σ^0_2 set $Y \geq_T \emptyset'$, there exists a Δ^0_2 real $X \in P$ such that $X' \equiv_T Y$.

We need the following result of Kucera [11].

Lemma 2. *Let P be a Π^0_1 class of positive measure. Then there is a Π^0_1 class $Q \subseteq P$ and a computable function g such that $\mu(Q) > 0$ and, for all $e \in \omega$,*

$$Q \cap P_e \neq \emptyset \;\Rightarrow\; \mu(Q \cap P_e) \geq 2^{-g(e)}.$$

Theorem 2. *Let V be a pseudojump, $K \leq_T C$, and P be a Π^0_1 class with positive measure. Then there exist infinitely many $X \in P$ with $V^X \equiv_T X \oplus K \equiv_T C$.*

Proof. Let P be a Π^0_1 class of positive measure and let Q and g be given by Lemma 2. Now define the Π^0_1 relation E by

$$E = \{\langle e, \sigma\rangle : \sigma \in 2^{g(e)+1} \;\&\; I(\sigma) \cap Q \cap P_e \neq \emptyset\}.$$

Then we will define two functions λ and ρ, computable in K, which will provide distinct Π^0_1 subclasses $P_{\lambda(e)}$ and $P_{\rho(e)}$ of $P_e \cap Q$ when $P_e \cap Q \neq \emptyset$. That is, if $P_e \cap Q \neq \emptyset$, then $\mu(Q \cap P_e) \geq 2^{-g(e)}$ and hence $\{\sigma : E(e, \sigma)\}$ has at least two elements. Let $\sigma_{e,l}$ be the leftmost string such that $E(e, \sigma)$ and let

$$P_{\lambda(e)} = P_e \cap Q \cap I(\sigma_{e,l};$$

similarly, let $\sigma_{e,r}$ be the righttmost string such that $E(e, \sigma)$ and let

$$P_{\rho(e)} = P_e \cap Q \cap I(\sigma_{e,r}.$$

We modify the proof of Theorem 1 above, using the functions λ and ρ in place of the splitting function g.

Fix e such that $V^X = W^X_e$ and let

$$U_a = \left\{\, \sigma \in \{0,1\}^* \,:\, \phi^{\sigma}_{e,|\sigma|}(a)\!\uparrow \,\right\},$$

just as in Theorem 1 as above. Now define the sequences $\{R_n : n < \omega\}$ and $\{Q_n : n < \omega\}$ of Π^0_1 classes as follows. Let $Q_0 = Q = P_c$ and let

$$R_n = \begin{cases} Q_n \cap [U_n], & \text{if } Q_n \cap [U_n] \text{ is nonempty,} \\ Q_n, & \text{otherwise.} \end{cases}$$

Let $R_n = P_{r(n)}$. By the construction, R_n is a nonempty subset of Q, so that $P_{\lambda(r(n))}$ and $P_{\rho(r(n))}$ are both nonempty subsets of Q. Then define

$$Q_{n+1} = \begin{cases} P_{\lambda(r(n))}, & \text{if } n \in C, \\ P_{\rho((r(n))}, & \text{otherwise.} \end{cases}$$

Let X be any element of $\cap_n Q_n$. Recall that, for any Π^0_1 class P, T_P is the computable tree such that $P = [T_P]$.

First, we claim $V^X \leq_T C$. Note that $r(0) = c$ if and only if $0 \in V^X$, which happens if and only if the computable tree $T_{Q_0} \cap U_0$ is finite. This can be determined by $K \leq_T C$; hence C can determine $V^X(0)$ and the index $r(0)$ of R_0. Now, if $0 \in C$, $Q_1 = P_{\lambda(r(0)}$, and, if $0 \notin C$, $Q_1 = P_{\rho(r(0))}$. Since λ and ρ are computable in K, it follows that C can compute an index $q(1)$ for Q_1. Using this as a model, it follows by a routine induction that functions r and q, giving the indices of R_n and Q_n respectively for all n, are computable in C. Since $n \in V^X$ if and only if $T_{Q_n} \cap U_n$ is finite, and this fact is computable in $K \leq_T C$, $V^X \leq_T C$.

Next, we claim $V^X \leq_T X \oplus K$. The only place where we use C rather than K in the previous argument is to determine the index of Q_{n+1} from $r(n)$. We can instead use K to compute the strings $\sigma_{r(n),l}$ and $\sigma_{r(n),r}$ and then use X to determine which of these is an initial segment of X. If $\sigma_{r(n),l} \sqsubset X$, then $X \in I(\lambda(r(n)))$ and $Q_{n+1} = R_n \cap I(\lambda(r(n)))$; if $\sigma_{r(n),r} \sqsubset X$, then $X \in I(\rho((r(n)))$ and $Q_{n+1} = R_n \cap \rho(g(r(n)))$. Since λ and ρ are computable in K, it follows that $q(n{+}1)$ is computable in $X \oplus K$. It follows as above that V^X is computable from $X \oplus K$.

Thus we have $V^X \leq_T X \oplus K \leq_T C$. It remains to show that $C \leq_T V^X$ and for this we show that the construction is computable from V^X. Given an index q for Q_n, we can compute an index for $Q_n \cap [U_n]$ and we know that $R_n = Q_n \cap [U_n]$ if $n \in V^X$, and $R_n = Q_n$ otherwise. Thus we can obtain an index for R_n from V^X and an index for Q_n. Next, given the index $r(n)$ for R_n, we can compute $g(r(n))$ and $\sigma = X \lceil g(r(n)) + 1$. Then we can use the formula $Q_{n+1} = R_n \cap Q \cap I(\sigma)$, to obtain an index for Q_{n+1}. Furthermore, we know that σ is either the leftmost or the rightmost string in T_{R_n} of length $g(r(n)) + 1$ and we can determine which of these cases holds as follows. Enumerate the complement of $T_{R(n)}$ until either (i) all strings of length $g(r(n)) + 1$ to the left of σ have been enumerated or (ii) all strings of length $g(r(n)) + 1$ to the right of σ have been enumerated; note that only one of these is possible since there are at least two strings in T_{R_n} of length $g(r(n)) + 1$. In case (i), we conclude that $\sigma = \sigma_{r(n),l}$ and hence $n \in C$. In case (ii), $\sigma = \sigma_{r(n),r}$ and $n \notin C$.

To obtain infinitely many X with $V^X \equiv_T X \oplus K \equiv_T C$,

Now for any σ such that $\mu(P \cap I(\sigma)) > 0$, it follows that $P \cap I(\sigma)$ contains some X with $V^X \equiv_T X \oplus K \equiv_T C$. But given $\mu(P) > 0$, we can easily find an infinite sequence of disjoint intervals $I(\sigma_k)$ such that $P \cap I(\sigma_k) > 0$ for each k and thus an infinite sequence $X_k \in I(\sigma_k) \cap P$ with the desired properties. $\square$

We conjecture that if $C = S'$, then it is possible to obtain $X \leq_T S \oplus K$ in Theorem 2.

We can obtain several corollaries to Theorem 2. Taking V to be the trivial operator in Theorem 1 actually shows the following:

Corollary 3. *Any Π^0_1 class of positive measure contains sets of every degree above $\emptyset'$.*

Taking V to be the jump, we have the following:

Corollary 4. *If P has positive measure and $\emptyset' \leq_T C$, then there exists $X \in P$ so that $X' \equiv_T C$.*

It is well-known that there exists a Π^0_1 class P with positive measure containing only 1-random reals. Hence we can also obtain new results on jump inversion by 1-random reals.

Corollary 5. *Let V be a pseudojump, and let $K \leq_T C$. Then there is a 1-random real X with $V^X \equiv_T X \oplus K \equiv_T C$.*

Taking V to be the jump, we have another proof of the result that 1-random reals can have all possible jumps. Taking V to be the trivial operator, we have the following.

Corollary 6. *There exist 1-random reals of every degree above $\emptyset'$.*

4 Images of Π^0_1 classes under c.e. operators

Although the class Q constructed in Theorem 1 is not a Π^0_1 class, it is a strong Π^0_2 class with the property that $\{V^X : X \in Q\}$ is a singleton, since, given any $X \in Q$, for every n, $n \in V^X$ if and only if $T_{Q_n} \cap U_n$ is finite. It seems natural to consider the question of a Π^0_1 class P where V^X is unique for $X \in P$. A classical result is that if $P = \{X\}$ itself is a singleton, then X is computable. Of course, for an CEA operator, J_e, if $J_e(X) = J_e(Y)$, then $X = Y$. In fact, there seems to be no good reason at the outset to restrict ourselves to pseudojumps, so we consider the more general case of c.e. operators W^X_e without demanding that $X \leq_T W^X_e$. Of course, it is easy to give examples where the Π^0_1 set is not a singleton, since W^X_e might not consult the oracle X at all, or more generally, only consult some fixed finite amount of the oracle: for instance, if

$$W^X_e = \begin{cases} \omega, & \text{if } 0 \in X \\ X, & \text{otherwise,} \end{cases}$$

the the class of all sets containing 0 is an uncountable example. The following proposition shows that any example is bound in some sense to make an inessential use of the oracle X, as least as far as the Π^0_1 class is concerned.

Proposition 7. *If P is a Π^0_1 class and $\left\{ W_e^X : X \in P \right\}$ is a singleton, then the unique W_e^X for $X \in P$ is a c.e. set.*

Proof. Fix a computable tree T such that $P = [T]$. We claim

$$a \in V^X \text{ if and only if } (\exists n)[(\forall \sigma \in \{0,1\}^n \cap T) a \in V^\sigma].$$

Suppose first that $a \in V^X$ for all $X \in P$. Then by compactness, there exists m such that $a \in V^{X \restriction m}$ for all $X \in P$. Let $S = \{\sigma \in \{0,1\}^m : P \cap I(\sigma) \neq \emptyset\} = \{X \restriction m : X \in P\}$. For $\sigma \in \{0,1\}^m - S$, T contains only finitely many extensions of σ. Thus we can find $n > m$ such that $\tau \restriction m \in S$ for all $\tau \in \{0,1\}^n \cap T$. This n satisfies the formula above.

Next suppose that n exists as in the formula. Then for every $X \in P$, $a \in V^{X \restriction n}$ and therefore $a \in V^X$. $\square$

When W_e^X is actually a pseudojump, we can say more, using a classical fact due to Kreisel.

Proposition 8. *Suppose P is a Π^0_1 class and for all X, $X \leq_T W_e^X$. If $\left\{ W_e^X : X \in P \right\}$ is countable, then $\left\{ W_e^X : X \in P \right\}$ has a c.e. member.*

Proof. Since $X \leq_T W_e^X$ for all X, and $\left\{ W_e^X : X \in P \right\}$ is countable, P must itself be countable. Then by Kreisel [10], P has a computable member, R. Since W_e^R is a c.e. member of $\left\{ W_e^X : X \in P \right\}$, the result follows immediately. $\square$

Corollary 9. *Suppose P is a Π^0_1 class and for all X, $X \leq_T W_e^X$. If $\left\{ W_e^X : X \in P \right\}$ all lies in one Turing degree $\mathbf{w}$, then $\mathbf{w}$ is a c.e. degree.*

The same considerations used in the proof of Proposition 8 also yield the following when e involves the construction of an incomplete c.e. degree.

Proposition 10. *Suppose P is a Π^0_1 class and for all X, $X \leq_T W_e^X$ and that $\left\{ W_e^X : X \in P \right\}$ all lies in one Turing degree $\mathbf{w}$. If for all computable R, $W_e^R <_T K$, then $\mathbf{w} < \mathbf{0}'$.*

For the other extreme, suppose that V^X is Turing incomparable with V^Y for all $X \neq Y$ in P. It was also shown in [7] that there exist Π^0_1 classes containing continuum many elements, with each pair Turing incomparable. This will serve as an example with $V^X = X$.

Of course if $V^X = X'$, then any Π^0_1 class Q must contain X with $V^X \equiv_T K$ and therefore, if nontrivial, Q must contain distinct X, Y with $V^X \equiv_T K \equiv_T V^Y$.

Proposition 11. *Let W^X denote either W_e^X or $X \oplus W_e^X$ and suppose that P is an infinite Π^0_1 class such that W^X and W^Y are Turing incomparable for any $X, Y \in P$. Then there is no $X \in P$ such that $K \leq_T W^X$.*

Proof. Suppose by way of contradiction that $K \leq_T W^X$ for some $X \in P$. Since P is infinite, there is some $Y \in P$ with $Y \neq X$. Let n be the least such that $X(n) \neq Y(n)$ and let $Q = P \cap I(Y \restriction n+1)$. By Proposition 1, there exists $Z \in Q$ with $W^Z \leq_T K \leq_T V^X$. $\square$

5 Using Π^0_1 classes to define pseudojumps

Finally, we observe that Π^0_1 classes may be used to define the jump and also general c.e. operators.

Proposition 12. *Let* $\{ P_e : e \in \omega \}$ *be the standard enumeration of* Π^0_1 *classes; then, for any set* X*,* $\{ e : X \in P_e \} \equiv_T X'$*.*

Proof. Let $W^X = \{ e : X \in P_e \}$. Then $W^X \leq_T X'$ since

$$e \in W^X \text{ if and only if } (\forall n) X \restriction n \notin W_e.$$

For the completeness, use the s-m-n theorem to define a computable function f such that $P_{f(e)} = \{ X : \phi^X_e(e)\uparrow \}$. Then

$$e \in X' \text{ if and only if } f(e) \notin W^X$$

gives a reduction of X' to W^X.

□

One can define more general operators using Π^0_1 classes as follows. Given $X \in 2^{\mathbb{N}}$, let $\pi_i(X) = \{ j : \langle i, j \rangle \in X \}$. For a Π^0_1 class P, let $\pi_i(P) = \{ \pi_i(X) : X \in P \}$, the projection of P onto the ith coordinate.

Then let

$$V^{\pi,X}_e = \{i : X \in \pi_i(P_e)\}.$$

Since P_e is Π^0_1, $\pi_i(P_e)$ is a Σ^1_1 set; however, $V^{\pi,X}_e$ is itself merely $\Pi^0_1(X)$. To see this, let R be a primitive recursive predicate so that $Y \in P_e$ if and only if $\forall z, R(Y \restriction z)$. For any $\sigma \in {}^{<\omega}2$, let

$$\sigma^{[i]}(j) = \sigma(\langle i, j \rangle).$$

Let $T = \{ \sigma \in {}^{<\omega}2 : R(\sigma) \wedge \sigma^{[i]} = X \restriction i \}$. T is clearly an X-computable tree, and $[T]$ is empty if and only if $i \notin V^{\pi,X}_e$. Since this is a $\Sigma^0_1(X)$ question, $V^{\pi,X}_e$ is $\Pi^0_1(X)$. Strictly speaking, $V^{\pi,X}_e$ defined this way is not a pseudojump, since it produces the complement of a $\Sigma^0_1(X)$ set. So, of course, if an operator can be defined in both ways, it is uniformly trivial, since it only gives sets computable in the operand. Note that $V^{\pi,X}_e \equiv_T X'$ when P_e is a particular Medvedev complete class, namely, one such that $\pi_i(P)$ runs over all Π^0_1 classes.

The natural question that arises is whether (up to complementation) every c.e. operator can be expressed in this form. If $P = \emptyset$, then $\overline{\{i : X \in \pi_i(P_e)\}} = \omega$. Once we rule out this trivial case, there is a natural condition that answers this question.

Proposition 13. *Let* $e \in \omega$ *and suppose there exists some* X *so that* $W^X_e \neq \omega$*. Then the following are equivalent:*

(a) For every $j \in \omega$ there exists some Y such that $\phi_e^Y(j)\uparrow$.

(b) There exists a nonempty Π_1^0 class P such that for all X, $\overline{W_e^X} = V_e^{\pi,X}$.

Proof. First, suppose e is such that for every $j \in \omega$ there exists some Y such that $\phi_e^Y(j)\uparrow$. Let

$$T = \{ \sigma : \forall i\, \phi_e^{\sigma^i}(i)\uparrow \},$$

where $\sigma^i(j)$ is defined as above. We claim $X \in \pi_i([T])$ if and only if $i \notin W_e^X$.

Now suppose $X \in \pi_i([T])$, and let $Y \in [T]$ such that $\pi_i(Y) = X$. Now, if $\phi_e^X(i)\downarrow$, then $\exists \sigma \subset X$ so that $\phi_e^\sigma(i)\downarrow$. Let $\tau = Y \upharpoonright |\sigma|^2$. Since $\sigma \subseteq \tau^i$, $\phi_e^{\tau^i}(i)\downarrow$. But then $\tau \notin T$, hence $Y \notin [T]$, a contradiction. Hence, If $X \in \pi_i([T])$, $i \notin W_e^X$.

On the other hand, suppose $i \notin W_e^X$. By assumption, $\forall j\, \exists Y\, \phi_e^Y(j)\uparrow$. For each j, let Y_j be such a set. Let

$$Z(\langle j, w\rangle) = \begin{cases} X(w), & \text{if } j = i, \text{ and} \\ Y_j(w), & \text{otherwise.} \end{cases}$$

Then $\forall j\, \phi_e^{\pi_j(Z)}(j)\uparrow$, so $Z \in [T]$. Since $\pi_i(Z) = X$, $X \in \pi_i([T])$. This shows $(a) \Rightarrow (b)$.

For the other direction, suppose that there is some j such that for every Y $\phi_e^Y(j)\downarrow$. Then if P is such that for all X, $\overline{W_e^X} = \{ i : X \in \pi_i(P) \}$, we must have for every X, $X \notin \pi_i(P)$. Hence $P = \emptyset$. But then, for every X and i, $X \notin \pi_i(P)$, so that for every X, $W_e^X = \omega$, a contradiction. ☐

That this limitation is really just an inessential feature of using projections, can be seen from the following observation:

Proposition 14. *There exist computable functions f and g such that*

1. *for every $e \in \omega$, $W_e =^* W_{f(e)}$ and $W_e =^* W_{g(e)}$;*
2. *for every e and j, there exists Y such that $\phi_{f(e)}^Y(j)\uparrow$;*
3. *for every e there exists j, such that for every Y $\phi_{g(e)}^Y(j)\downarrow$.*

Proof. For any oracle X, let

$$\phi_{f(e)}^X(j) = \begin{cases} \uparrow, & \text{if } X = \{j\}, \text{ and} \\ \phi_e^X(j), & \text{otherwise;} \end{cases}$$

and let

$$\phi_{g(e)}^X(j) = \begin{cases} 0, & \text{if } j = 0, \text{ and} \\ \phi_e^X(j), & \text{otherwise.} \end{cases}$$

The condition $X = \{j\}$ is $\Pi_1^0(X)$, so $\phi_{f(e)}^X$ is X-computable. It is straightforward to check that f and g are otherwise as required. ☐

Thus any c.e. operator is almost equivalent both to one that can be expressed in terms of projections of effectively closed sets and one that cannot be so expressed.

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