Embedding the Diamond Lattice in the c.e. *tt*-Degrees with Superhigh Atoms

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Abstract. The notion of superhigh computably enumerable (c.e.) degrees was first introduced by Mohrherr in [7], where she proved the existence of incomplete superhigh c.e. degrees, and high, but not superhigh, c.e. degrees. Recent research shows that the notion of superhighness is closely related to algorithmic randomness and effective measure theory. Jockusch and Mohrherr proved in [4] that the diamond lattice can be embedded into the c.e. *tt*-degrees preserving 0 and 1 and that the two atoms can be low. In this paper, we prove that the two atoms in such embeddings can also be superhigh.

1 Introduction

Lachlan proved in 1966 in [5] the classical Non-Diamond Theorem: no diamond can be embedded in the c.e. Turing degrees preserving both 0 and 1. However, Cooper showed that such a diamond can be embedded into the Δ^0_2 degrees if we do not require that the atoms be c.e. [1]. Later, Epstein showed that both atoms can be made low and that both atoms can be made high [3], and Downey proved in [2] that both atoms can be d.c.e. degrees, giving an extremely sharp result in terms of the Ershov hierarchy.

Alternately, we can consider the possibility of constructing a diamond preserving 0 and 1 if we consider a stronger reducibility. Since the proof of Lachlan's Non-Diamond Theorem holds in the c.e. *wtt*-degrees as well, no such diamond exists in the c.e. *wtt*-degrees. However, Jockusch and Mohrherr showed in [4] that the diamond lattice can be embedded into the c.e. *tt*-degrees preserving 0 and 1 and, furthermore, that the two atoms can be low. In this paper, we present a proof that such a diamond can be embedded into the c.e. *tt*-degrees in such a way that both atoms are superhigh.

The notion of superhigh c.e. degrees was first introduced by Mohrherr in [7], where a computably enumerable set A is defined to be *superhigh* if $A' \equiv_{tt} \emptyset''$.

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In the same paper, Mohrherr proved the existence of incomplete superhigh c.e. degrees and also the existence of high, but not superhigh, c.e. degrees. More recently, Ng has shown in [8] that there is a minimal pair of superhigh c.e. degrees. Recent research in computability theory shows that the notion of superhighness is closely related to algorithmic randomness and effective measure theory. For instance, Simpson showed that uniformly almost everywhere dominating degrees are all superhigh [9] (the uniformly almost everywhere dominating degrees are all high follows from the characterization of highness via domination due to Martin), and Kjös-Hanssen, Miller and Solomon showed that the uniformly almost everywhere dominating degrees are exactly the degrees containing a set A such that $\emptyset'$ is K-trivial relative to A.

Our theorem is stated as follows.

Theorem 1. *There are superhigh computably enumerable sets A and B such that $\mathbf{0}$, $deg_{tt}(A)$, $deg_{tt}(B)$, and $\mathbf{0}'_{tt}$ form a diamond in the computably enumerable tt-degrees.*

Our construction differs from Jockusch and Mohrherr's in several important ways. Jockusch and Mohrherr's construction involves only a finite injury argument, while ours involves an infinite injury argument, which is necessary to make A and B superhigh. Due to this, our sets A and B will not have some of the nice properties that Jockusch and Mohrherr's do. For instance, they were able to build their atoms A and B with $A \cup B = K$, guaranteeing that $K \equiv_{tt} A \cup B$ in a very obvious way. In our construction, the superhighness strategies will force us to enumerate elements into A and B from time to time to maintain our computations that witness $A' \geq_{tt} TOT$ and $B' \geq_{tt} TOT$, where $TOT = \{e : \varphi_e \text{ is total}\}$ is a Π_2-complete set. To ensure that $K \leq_{tt} A \oplus B$, we dedicate the numbers of the form $\langle x, 0\rangle$ to meeting this requirement. This allows us to replace Jockusch and Mohrherr's conclusion that $x \in K$ if and only if $x \in A \cup B$ by the slightly more complicated conclusion that $x \in K$ if and only if $\langle x, 0\rangle \in A \cup B$. Again, for the consistency between the superhighness strategies and the minimal pair strategies, we need to be extremely careful when we switch from one outcome to another one.

Our notations and terminologies are standard and generally follow Soare [10]. Let φ_e and Φ_e^A be the e-th partial computable function and the e-th A-partial computable function, respectively. In particular, if $\varphi_e(x)\downarrow$, then $[e](x)$ denotes the truth table with index $\varphi_e(x)$ in some effective enumeration of all truth tables, denoted as $\tau_{\varphi_e(x)}$, and $|[e](x)|$ denotes the length of this truth table. For any set A, $[e]^A(x)$ is 0 or 1 depending on whether or not A satisfies the truth table condition with index $\varphi_e(x)$ (denoted by $A \models [e](x)$ if $[e](x) = 1$, otherwise, $A \not\models [e](x)$). Given two sets A and B, we say that $A \leq_{tt} B$ iff there is an e with φ_e total such that for all x, $[e]^B(x) = A(x)$. When we choose a *fresh* number as a γ-use or a δ-use at stage s, this number is the least number bigger than the corresponding restraint that is not of the form $\langle x, 0\rangle$.

2 Requirements and Basic Strategies

To prove Theorem 1, we will construct two c.e. sets A and B such that both of them are superhigh, K is truth-table reducible to $A \oplus B$, and the tt-degrees of A and B form a minimal pair in the tt-degrees. A and B will satisfy the following requirements:

$\mathcal{P}$: $K \leq_{tt} A \oplus B$;
$\mathcal{S}^A$: $TOT \leq_{tt} A'$;
$\mathcal{S}^B$: $TOT \leq_{tt} B'$;
$\mathcal{N}_{i,j}$: $[i]^A = [j]^B = f$ total $\Rightarrow f$ is computable;

Recall that $TOT = \{e : \varphi_e \text{ is total}\}$ is a Π^0_2-complete set. Therefore, if $\mathcal{S}^A$ and $\mathcal{S}^B$ are satisfied, then A and B will both be superhigh.

2.1 The $\mathcal{P}$-Strategy

To satisfy the requirement $\mathcal{P}$, we simply code K into $A \oplus B$. We will fix a computable enumeration of K such that at each odd stage s, exactly one number, k_s, enters K. At each odd stage s, we will enumerate $\langle k_s, 0\rangle$ into A, B, or both. We will decide which of these sets to enumerate $\langle k_s, 0\rangle$ into based on the actions of the minimal pair strategies $\mathcal{N}_{i,j}$. If $k \notin K$, then numbers of the form $\langle k, 0\rangle$ will never be enumerated into A and B. It is obvious that we will have the equality $K = \{k : \langle k, 0\rangle \in A \cup B\}$, and hence $K \leq_{tt} A \oplus B$.

The $\mathcal{P}$-requirement is global, so we do not need to place it on the construction tree.

2.2 An $\mathcal{S}^A_e$-Strategy

To make A superhigh, instead of giving a truth-table reduction from TOT to A' explicitly, we will construct a binary functional $\Gamma^A(e, x)$ such that for all $e \in \omega$,

$$\mathrm{TOT}(e) = \lim_{x\to\infty} \Gamma^A(e, x)$$

with $|\{x : \Gamma^A(e, x) \neq \Gamma^A(e, x + 1)\}|$ bounded by a computable function h, which will ensure that TOT $\leq_{tt} A'$. (In the case of B, we will construct a binary functional $\Delta^B(e, y)$ with use $\delta(e, y)$ satisfying a similar requirement.) The crucial point is to find this computable bounding function h.

As usual, $\mathcal{S}^A$ is divided into infinitely many substrategies $\mathcal{S}^A_e$, $e \in \omega$, each of which is responsible for the definition of $\Gamma^A(e, x)$ for $x \in \omega$, and has two outcomes, ∞ (a Π^0_2-outcome) and f (a Σ^0_2-outcome), where ∞ denotes the guess that φ_e is total and f denotes the guess that φ_e is not total. The main idea is that all the $\mathcal{S}^A_e$ strategies (they will be arranged on a single level on the construction tree) work for the definition of $\Gamma^A(e, x)$, $x \in \omega$, jointly, and the one on the true path defines $\Gamma^A(e, x)$ for almost all x such that $\lim_{x\to\infty} \Gamma^A(e, x)$ exists and equals to $TOT(e)$.

Let β be an $\mathcal{S}_e^A$-strategy on the priority tree. As usual, we have the following standard definition of length agreement function:

$$l(\beta, s) = \max\{\ x < s : s \text{ is a } \beta\text{-stage and } \varphi_e(y)[s]\downarrow \quad \text{for all} \quad y < x\};$$

$$m(\beta, s) = \max\{\ l(\beta, t) : t < s \text{ is a } \beta\text{-stage}\}.$$

Say that s is a β-expansionary stage if $s = 0$ or $l(\beta, s) > m(\beta, s)$.

Let s be a β-stage. If s is a β-expansionary stage, then we believe that φ_e is total, and for those $\Gamma^A(e, x)$ either defined by lower priority strategies, or defined by β itself, but with value 0, we undefine them by enumerating the corresponding $\gamma(e, x)$ into A and then define $\Gamma^A(e, y)$ to be 1 for the least y such that $\Gamma^A(e, y)$ is undefined. If s is not an β-expansionary stage, then we believe that φ_e is not total, and again, we undefine those $\Gamma^A(e, x)$ defined by lower priority strategies by enumerating the corresponding $\gamma(e, x)$ into A and then define $\Gamma^A(e, y)$ to be 0 for the least y with $\Gamma^A(e, y)$ not defined. Thus, if there are infinitely many β-expansionary stages (so φ_e is total, $e \in$ TOT, and ∞ is the true outcome of β), then $\Gamma^A(e, x)$ is defined as 1 for almost all $x \in \omega$. On the other hand, if there are only finitely many β-expansionary stages (so φ_e is not total, $e \notin$ TOT, and f is the true outcome of β), then $\Gamma^A(e, x)$ is defined as 0 for almost all $x \in \omega$.

Thus, for a fixed $\mathcal{S}_e^A$-strategy β on the construction tree, β will attempt to redefine $\Gamma^A(e, x)$ for almost all x. The only γ-uses which it will not be allowed to enumerate into A are (a) some γ-uses are prevented from being enumerated into A by higher priority strategies (when a disagreement is produced), or (b) $\Gamma^A(e, x)$ is defined by another $\mathcal{S}_e^A$-strategy with higher priority. In particular, if β is the $\mathcal{S}_e^A$-strategy on the true path, then there are only finitely many strategies with higher priority that can be visited during the whole construction, and hence β can succeed in defining $\Gamma^A(e, x)$ for almost all x.

Suppose that $\mathcal{S}_e^A$ is assigned to nodes on level n. We will see that $|\{x : \Gamma^A(e, x) \neq \Gamma^A(e, x+1)\}| \leq 2^{3^n+1}$. To see this, note that (a) above can happen at most 2^{3^n} times, as there are at most 3^n many strategies with length less than n, and each time when one of them produces (not preserves) a disagreement, a restraint is set, preventing α from rectifying $\Gamma^A(e, x)$ for some x. Note that after an $\mathcal{N}$-strategy α (see below) produces a disagreement, say at stage s, whenever α requires us to preserve this disagreement, all the strategies with lower priority will be initialized, and at the same time, all of the γ-uses and δ-uses defined after stage s will be enumerated into A and B respectively (one by one, as pointed out above, for the sake of the $\mathcal{N}$-strategies with priority higher than α). It is crucial for us to ensure that TOT is truth-table reducible to A' and B', as we will discuss below.

Here, when β is initialized by a strategy with higher priority with length $\geq n$, an $\mathcal{S}_e^A$-strategy β' on the left of β is visited, and β' takes the responsibility of rectifying $\Gamma^A(e, x)$ for some x, which can lead to an equality between $\Gamma^A(e, x)$ and $\Gamma^A(e, x+1)$. Thus, (b) can happen at most 3^n many times. In total, the number of those x such that β cannot rectify $\Gamma^A(e, x)$ is at most 2^{3^n+1}, which ensures that $Tot \leq_{tt} A'$, where the corresponding bounding function h is given by $h(e) = 2^{3^{3e}+1}$.

We remark here that as a bounding function, h is not tight, but it is enough to show that TOT is truth-table reducible to A', as we want.

2.3 An $\mathcal{N}_{i,j}$-Strategy

Recall that if $[i]$ is a *tt*-reduction, then for any oracle $X \subseteq \omega$ and any input x, $[i]^X(x)$ converges. The computation $[i]^X(x)$ can be injured at most finitely many times due to the enumeration of numbers less than or equal to $|\tau_{\varphi_i(x)}|$ into X in our construction.

For the requirement $\mathcal{N}_{i,j}$, we apply the diagonalization argument introduced by Jockusch and Mohrherr in [4]. That is, once we see a disagreement between $[i]^A$ and $[j]^B$, we will preserve it forever to make $[i]^A \neq [j]^B$. On the other hand, if $[i]^A$ and $[j]^B$ are equal and total, then we will ensure that they are computable.

Given values for A_s and B_s at stage s, we will define A_{s+1} and B_{s+1} at stage $s+1$ by possibly enumerating into them. Furthermore, if we know that $[i]^A$ and $[j]^B$ differ at k at stage s, we will have to preserve this disagreement at stage $s+1$. This is achieved by the following. Let n be a number we want to put into $A_{s+1} \cup B_{s+1}$. There are two cases.

(1) Our number n is of the form $\langle x, 0\rangle$ for some x. Then n is enumerated into A, B, or both for the sake of the requirement $\mathcal{P}$. There are three subcases.

Subcase 1: If $[i]^{A_s}(k) = [i]^{A_s \cup \{n\}}(k)$, then n will be enumerated into A but not into B. The disagreement is preserved as well.

Subcase 2: If Subcase 1 does not apply but $[j]^{B_s}(k) = [j]^{B_s \cup \{n\}}(k)$, then n is enumerated into B but not into A. As in Case 1, the disagreement is preserved.

Subcase 3: If $[i]^{A_s}(k) \neq [i]^{A_s \cup \{n\}}(k)$ and $[j]^{B_s}(k) \neq [j]^{B_s \cup \{n\}}(k)$, then n is enumerated into both A and B. In this case, the disagreement is again preserved, as both values are changed.

Note that once one subcase above applies, then we initialize all the strategies with lower priority to avoid conflict among the $\mathcal{N}$-strategies — obviously, such initializations can happen at most finitely often. We need to be careful here when more $\mathcal{N}$-strategies are considered. It can happen that if we decide to enumerate into A, B, or both, we also need to take care of those $\mathcal{N}$-strategies with higher priority, say $\mathcal{N}_{i',j'}$, as we need to avoid the following situation: according to the $\mathcal{N}_{i,j}$-strategy, at stage s_1, a number n_1 is enumerated into A, and at stage s_2, a number n_2 is enumerated into B (corresponding to Subcases 1 and 2, respectively), and such enumerations change $[i']^A(m)$ and $[j']^B(m)$, though separately, and at the next $\mathcal{N}_{i',j'}$-expansionary stage, we may have $[i']^A(m) = [j']^B(m)$, which is different from its original value — $\mathcal{N}_{i',j'}$ is injured.

With this in mind, when we see that an $\mathcal{N}_{i,j}$-strategy wants to enumerate a number into A (or B, or both), instead of enumerating it immediately, we first check whether such an enumeration into A can lead to a disagreement between $[i']^A$ and $[j']^B$. If not, then we just work as described above (in Subcase 3, we now enumerate n into B and check whether this enumeration into B can lead to a disagreement for $\mathcal{N}_{i',j'}$ — here n is enumerated into A and B separately). Otherwise, we start to preserve this disagreement to satisfy $\mathcal{N}_{i',j'}$ — the $\mathcal{N}_{i,j}$ considered above is initialized, and again, even if Subcase 3 applies, we do not enumerate n into B.

The $\mathcal{N}$-strategies are arranged linearly according to priority, and each time $\mathcal{P}$ decides to act it checks for the highest priority $\mathcal{N}$-strategy for which the enumeration of $\langle k, 0\rangle$ into A or B will change an $\mathcal{N}$-computation. We then act for $\mathcal{N}$ as in subcases 1-3 above. This clearly injures an $\mathcal{N}'$-strategy of lower priority and it will need to be initialized, but it is easy to see that each $\mathcal{N}'$ is injured in this way by the global $\mathcal{P}$ only finitely often.

(2) Our number n is a number chosen by an $\mathcal{S}_e^A$-strategy or an $\mathcal{S}_e^B$-strategy. Without loss of generality, suppose that n is selected by an $\mathcal{S}_e^A$-strategy and we want to put it into A. As in the standard construction of high sets, we only consider believable computations; for instance, $[i]^A(m)$. Therefore, when we see $[i]^A$ and $[j]^B$, if this $\mathcal{S}_e^A$-strategy has higher priority than $\mathcal{N}_{i,j}$, then the enumeration of n into A does not affect the computation $[i]^A(m)$. We will have more discussion on this soon.

An $\mathcal{N}_{i,j}$-strategy has three outcomes: ∞, f and d, where ∞ denotes that there are infinitely many expansionary stages, f denotes that there are only finitely many expansionary stages, but no disagreement is produced, and d denotes that a disagreement between $[i]^A$ and $[j]^B$ is produced and preserved successfully.

2.4 More on Interactions among Strategies

We have seen some interactions between the $\mathcal{P}$-strategy and the $\mathcal{N}$-strategies. Now we describe the interactions between the $\mathcal{N}$-strategies, the $\mathcal{S}$-strategies, and the $\mathcal{P}$-strategy.

Assume that α is an $\mathcal{N}_{i,j}$-strategy, β is an $\mathcal{S}_e^A$-strategy, and ζ is an $\mathcal{S}_{e'}^B$-strategy with $\beta^\frown\infty \subseteq \zeta^\frown\infty \subseteq \alpha$. The following may happen: at a stage s, a disagreement between $[i]^A$ and $[j]^B$ appears at α, so α wants to preserve this disagreement by initializing all strategies with lower priority. However, this disagreement can be destroyed by β and ζ, as they may enumerate small γ-uses and δ-uses into A and B separately. To avoid this, we only use α-believable computations, a standard technique in the construction of high degrees.

Definition 1. *Let α be an $\mathcal{N}_{i,j}$-strategy, and β be an $\mathcal{S}_e^A$-strategy with $\beta^\frown\infty \subseteq \alpha$.*

(1) A computation $[i]^{A_s}(m)$ is α-believable at β at stage s if for each x with $\gamma(e,x)[s]$ defined by β and less than the length of the truth-table of $[i](m)$, $\Gamma^{A_s}(e,x)[s]$ is equal to 1.

(2) A computation $[i]^{A_s}(m)$ is α-believable at stage s if it is α-believable at β at stage s for any $\mathcal{S}_e^A$-strategy β, $e \in \omega$, with $\beta^\frown\infty \subseteq \alpha$.

Similarly, we can define an α-believable computation $[j]^{B_s}(m)$.

We are ready to define an α-expansionary stage for an $\mathcal{N}_{i,j}$-strategy α.

Definition 2. *Let α be an $\mathcal{N}_{i,j}$-strategy. The length of agreement between $[i]^A$ and $[j]^B$ is defined as follows:*

$$l(\alpha, s) = \max\{x < s : \text{for all } y < x,\ [i]^A(y)[s] = [j]^B(y)[s] \text{ via } \alpha\text{-believable computations}\}.$$
$$m(\alpha, s) = \max\{l(\alpha, t) : t < s \text{ is an } \alpha\text{-stage}\}.$$

Say that a stage s is α-expansionary if $s = 0$ or $l(\alpha, s) > m(\alpha, s)$.

At an α-expansionary stage, before α is allowed to access outcome ∞, it has to clear every γ, δ-use in $F_\alpha^A \cup F_\alpha^B$, where F_α^A, F_α^B are the collections of γ, δ-uses defined by $\mathcal{S}$-strategies with priority lower than α after the last α-expansionary stage. We enumerate these uses one at a time into A or B respectively, until a disagreement is produced at some $\mathcal{N}'$-strategy $\beta \subset \alpha$. We then stop and do not access the nodes extending $\alpha^\frown\infty$ at this current stage. This is alright because a strong priority β has made permanent (subject to β's ability to protect this disagreement) progress on its basic strategy. We will refer to this enumeration process as an "*outcome-shifting enumeration process*" for simplicity. *So a tt-minimal pair strategy does does enumerate numbers into sets, which is completely difference from the minimal pair argument used in the c.e. Turing degrees.*

Now we consider the situation when β, an $\mathcal{S}_e^A$-strategy, changes its outcome from f to ∞ at a β-expansionary stage. Again, when β sees such an change of outcome, it also perform the outcome-shifting enumeration process by enumerating numbers into A and B as needed. That is, let s' be the last β-expansionary stage. Unlike the construction of high degrees, to make A and B superhigh, we need to enumerate all the γ-uses and δ-uses defined by strategies below outcome f, including those uses defined by β under the outcome f, between stages s' and s into A and B respectively. Again, these numbers cannot be enumerated into A and B simultaneously, as discussed above in the section on the $\mathcal{N}$-strategies, for the sake of $\mathcal{N}$-strategies with priority higher than β. Let F_β^A and F_β^B be the collections of these γ-uses and δ-uses respectively. We put the numbers in $F_\beta^A \cup F_\beta^B$ into A or B correspondingly, one by one, from the smallest to the largest, and whenever one number is enumerated, we reconsider the $\mathcal{N}$-strategies with higher priority to see whether a disagreement appears. Once such a disagreement appears at an $\mathcal{N}$-strategy, say α, we stop the enumeration as we need to satisfy α via this disagreement. In this case, β is injured. Note that β can be injured in this way only by those $\mathcal{N}$-strategies α such that $\alpha \subset \beta$.

2.5 Construction

First, we define the priority tree T and assign requirements to the nodes on T as follows. Suppose $\sigma \in T$. If $|\sigma| = 3e$, then σ is assigned to the $\mathcal{N}_{i,j}$-strategy such that $e = \langle i, j\rangle$. It has three possible outcomes: ∞, f, and d, with $\infty <_L f <_L d$. If $|\sigma| = 3e + 1$, then σ is assigned to the $\mathcal{S}_e^A$-strategy. If $|\sigma| = 3e + 2$, then σ is assigned to the $\mathcal{S}_e^B$-strategy. In the latter two cases, σ has two possible outcomes: ∞ and f, with $\infty <_L f$.

$\mathcal{P}$ is a global requirement, and we do not put it on the tree.

We assume that K is enumerated at odd stages. That is, we fix an enumeration $\{k_{2s+1}\}_{s\in\omega}$ of K such that at each odd stage $2s+1$, exactly one number, k_{2s+1}, is enumerated into K.

In the construction, we say that an $\mathcal{N}_{i,j}$-strategy α *sees a disagreement at k at a stage s* if $k \le s$, $[i]^{A_s}$ and $[j]^{B_s}$ agree on all arguments $\le k$, and one of the following cases applies:

(i) s is odd (k_s enters K and we need to put $\langle k_s, 0\rangle$ into $A \cup B$). In this case, either
   (1) $[i]^{A_s}(k) \ne [i]^{A_s \cup \{\langle k_s,0\rangle\}}(k)$,
   (2) $[j]^{B_s}(k) \ne [j]^{B_s \cup \{\langle k_s,0\rangle\}}(k)$, or
   (3) there is an $\mathcal{N}$-strategy $\alpha' \supset \alpha$ that attempts to preserve a disagreement, and the enumeration of $\langle k_s, 0\rangle$ into A or B or both (depending on α') and an one-by-one enumeration of elements of $F^A_{\alpha'} \cup F^B_{\alpha'}$ into A and B (in increasing order, as described in the $\mathcal{S}$-strategies) leads to either $[i]^A(k) \ne [i]^{A_s}(k)$ or $[j]^B(k) \ne [j]^{B_s}(k)$. Here, $F^A_{\alpha'}$ and $F^B_{\alpha'}$ are the finite collections of γ-uses and δ-uses defined below outcome $\alpha'^\frown d$ after the last stage α' that produces or preserves its disagreement.

   If (1) is true, then we enumerate $\langle k_s, 0\rangle$ into A. If (1) is not true but (2) is, then we enumerate $\langle k_s, 0\rangle$ into B. Otherwise, (3) is true, and we enumerate $\langle k_s, 0\rangle$ into A or B or both, according to α'. We also enumerate the corresponding numbers in $F^A_{\alpha'} \cup F^B_{\alpha'}$ into A and B respectively.

   As a consequence, a disagreement between $[i]^A(k)$ and $[j]^B(k)$ is produced, and α will preserve this disagreement forever unless it is initialized later.

(ii) s is even (s is a β-expansionary stage for some $\mathcal{S}$-strategy β).
   Let β be such a strategy, and let s' be the last β-expansionary stage. At stage s, to change its outcome from f to ∞, we need to enumerate all of the elements in F^A_β and F^B_β into A and B respectively one by one. Here, F^A_β and F^B_β are the finite collections of γ-uses and δ-uses defined below outcome $\beta^\frown f$, including those defined by β under the outcome f, after stage s'. Again, we enumerate these numbers into A and B in increasing order until we find that either $[i]^A(k) \ne [i]^{A_s}(k)$ or $[j]^B(k) \ne [j]^{B_s}(k)$ is true; that is, until a disagreement between $[i]^A(k)$ and $[j]^B(k)$ is produced. From now on, α will preserve this disagreement forever unless it is initialized later.

We recall that an $\mathcal{N}_{i,j}$-strategy α *preserves a disagreement at k at an odd stage s* if this disagreement was produced before and has been preserved so far (so $[i]^{A_s}(k) \ne [j]^{B_s}(k)$) and $\langle k_s, 0\rangle$ is less than one of the lengths of the truth-tables $[i](k)$ and $[j](k)$. Enumerating $\langle k_s, 0\rangle$ into $A \cup B$ causes one of the following to happen:

1. If $[i]^{A_s}(k) = [i]^{A_s \cup \{\langle k_s,0\rangle\}}(k)$, then $\langle k_s, 0\rangle$ is enumerated into A but not into B. Both values are preserved, and the disagreement is preserved as well.
2. If $[j]^{B_s}(k) = [j]^{B_s \cup \{\langle k_s,0\rangle\}}(k)$, then $\langle k_s, 0\rangle$ is enumerated into B but not into A. As in Case 1, the disagreement is preserved.

3. If $[i]^{A_s}(k) \neq [i]^{A_s \cup \{\langle k_s, 0\rangle\}}(k)$ and $[i]^{B_s}(k) \neq [i]^{B_s \cup \{\langle k_s, 0\rangle\}}(k)$, then $\langle k_s, 0\rangle$ is enumerated into both A and B. In this case, the disagreement is again preserved, as both values are changed.

Note that whenever α produces or preserves a disagreement in this manner, all the strategies below the outcome $\alpha^\frown d$ are initialized. Such initializations can happen at most finitely often.

Construction

Stage 0: Initialize all the nodes on T and set $A_0 = B_0 = \emptyset$. Let $\Gamma^A(e,x)[0]$ and $\Delta^B(e,x)[0]$ be undefined for each e and x.

Stage $s > 0$:

Case 1: *s is odd.* We will put $\langle k_s, 0\rangle$ into $A \cup B$ at this stage.

First check whether there is an $\mathcal{N}$-strategy that can produce a disagreement or needs to preserve a disagreement. Let α be the least such $\mathcal{N}$-strategy. Enumerate $\langle k_s, 0\rangle$ into A or B or both accordingly. Initialize all the strategies with lower priority.

Case 2: *s is even.* We define the approximation to the true path σ_s of length $\leq s$. Suppose that $\sigma_s\lceil u$ has been defined for $u \leq t$ and let ξ be $\sigma_s\lceil t$. We will define $\sigma_s(t)$. We have the following two subcases.

Subcase 1: ξ is an $\mathcal{N}_{i,j}$-strategy for some i and j. If ξ has produced a disagreement before and ξ has not been initialized since then, we let $\sigma_s(t) = d$. Otherwise, we check whether s is a ξ-expansionary stage. If not, then let $\sigma_s(t) = f$. If it is, then we start the outcome-shifting enumeration process to enumerate those γ-uses from F_ξ^A and δ-uses from F_ξ^B defined below the outcome $\xi^\frown f$ from the last ξ-expansionary stage into A and B respectively, one by one and in increasing order. At the same time, each time we enumerate such a number, we check whether there is an $\mathcal{N}$-strategy $\alpha \subset \xi$ that can produce a disagreement. If there is, then we stop the enumeration of F_ξ^A and F_ξ^B into A and B and let $\delta_s = \alpha$. Declare that α produces a disagreement at stage s, let $\sigma_s = \alpha$, and go to the 'defining' phase. If not, then after all numbers in $F_A \cup F_B$ have been enumerated, we let $\sigma_s(t) = \infty$ and go to the next substage.

Subcase 2: ξ is an $\mathcal{S}_e^A$-strategy or an $\mathcal{S}_e^B$-strategy for some e. If s is not a ξ-expansionary stage, let $\sigma_s(t) = f$ and go to the next substage. Otherwise, we start the outcome-shifting enumeration process as described in Subcase 1. Here F_ξ^A and F_ξ^B should also contain those γ-uses or δ-uses defined by ξ under the outcome f.

Defining Phase of stage s: For those $\mathcal{S}_e^A$-strategies β with $\beta^\frown\infty \subseteq \sigma_s$, find the least y such that $\Gamma^A(e,y)$ is currently not defined, define it as 1 and let the use $\gamma(e,y)$ be a fresh number, and for those $\mathcal{S}_e^A$-strategies β with $\beta^\frown f \subseteq \sigma_s$, find the least y such that $\Gamma^A(e,y)$ is currently not defined, define it as 0, and let the use $\gamma(e,y)$ be a fresh number. For those $\mathcal{S}_e^B$-strategies β, we define $\Delta^B(e,y)$ in the same way. Initialize all the strategies with lower priority than σ_s and go to the next stage.

Note that the enumeration of those γ-uses and δ-uses at substages into A and B ensures that those $\Gamma^A(e,x)$ and $\Delta^B(e,y)$ defined by those strategies with priority lower than σ_s are undefined.

This completes the construction.

Let $TP = \liminf_s \sigma_{2s}$ be the true path of the construction. We can first prove that TP is infinite and then verify that the construction given above satisfies all the requirements. Also it is obvious from the construction that

$$x \in K \iff \langle x, 0\rangle \in A \cup B,$$

and hence $K \leq_{tt} A \oplus B$.

This completes the proof of Theorem 1.

References

1. Cooper, S.B.: Degrees of unsolvability complementary between recursively enumerable degrees. I. Ann. Math. Logic 4, 31–73 (1972)
2. Downey, R.: D.r.e. degrees and the nondiamond theorem. Bull. London Math. Soc. 21, 43–50 (1989)
3. Epstein, R.L.: Minimal degrees of unsolvability and the full approximation construction. Mem. Amer. Math. Soc. 162(3) (1975)
4. Jockusch Jr., C.G., Mohrherr, J.: Embedding the diamond lattice in the recursively enumerable truth-table degrees. Proc. Amer. Math. Soc. 94, 123–128 (1985)
5. Lachlan, A.H.: Lower bounds for pairs of recursively enumerable degrees. Proc. London Math. Soc. 16, 537–569 (1966)
6. Martin, D.: Classes of recursively enumerable sets and degrees of unsolvability. Z. Math. Logik Grundlag. Math. 12, 295–310 (1966)
7. Mohrherr, J.: A refinement of low$_n$ and high$_n$ for the r.e. degrees. Z. Math. Logik Grundlag. Math. 32, 5–12 (1986)
8. Ng, K.M.: On Very High Degrees. Jour. Symb. Logic 73, 309–342 (2008)
9. Simpson, S.G.: Almost everywhere domination and superhighness. Math. Log. Q. 53, 462–482 (2007)
10. Soare, R.I.: Recursively enumerable sets and degrees. Springer, Heidelberg (1987)