Immunity and Non-Cupping for Closed Sets*

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Abstract

We extend the notion of immunity to closed sets and to Π^0_1 classes in particular in two ways: *immunity* meaning the corresponding tree has no infinite computable subset, and *tree-immunity* meaning it has no infinite computable sub*tree*. We separate these notions from each other and that of being *special*, and show separating classes for computably inseparable c.e. sets are immune and perfect thin classes are tree-immune. We define the notion of *prompt immunity* and construct a positive-measure promptly immune Π^0_1 class. We show that no immune-free Π^0_1 class P *cups* to the Medvedev complete class DNC of diagonally noncomputable sets, where P cups to Q in the Medvedv degrees of Π^0_1 classes if there is a class R such that the product $P \otimes R \equiv_M Q$. We characterize the interaction between (tree-)immunity and Medvedev meet and join, showing the (tree-)immune degrees form prime ideals in the Medvedev lattice. We show that every random closed set is immune and not small, and every small special class is immune.

Keywords: Computability, Π^0_1 Classes

1 Introduction

The notion of a simple c.e. set and the corresponding complementary notion of an immune co-c.e. set are fundamental to the study of c.e. sets and degrees. Together with variations and related notions such as *effectively immune*, *promptly simple*, *hyperimmune* and so forth, they permeate the classic text of R.I. Soare [25] and its updated version.

Many of the results on c.e. sets and degrees have found counterparts in the study of effectively closed sets (Π^0_1 classes). See the surveys [12, 13] for examples. In particular, hyperhyperimmune co-c.e. sets correspond to thin Π^0_1 classes [8, 11, 15] and hyperimmune co-c.e. sets correspond to several different notions including *smallness* studied by Binns [5, 6].

In this paper we consider the notion of immune sets as applied to Π^0_1 classes and closed sets in general. We work in $2^{\mathbb{N}}$ with the topology generated by basic clopen sets called *intervals*. For any $\sigma \in \{0,1\}^*$ the interval $I(\sigma)$ is $\{X : \sigma \prec X\}$, where $\prec$ means initial segment. Notation is standard; we note that λ denotes the empty string, $\sigma \upharpoonright n$ is the length-n initial segment of σ, and if $T \subseteq \{0,1\}^*$ is a tree (i.e., it is closed under initial segment), $[T] \subseteq 2^{\mathbb{N}}$ denotes the set of infinite paths through T. A node $\sigma \in T$ is a *leaf* of T if $\sigma^\frown i \notin T$ for any i. For any set $P \subseteq 2^{\mathbb{N}}$, we may define the tree $T_P = \{\sigma \in \{0,1\}^* : I(\sigma) \cap P \neq \emptyset\}$; the closed sets $P \subseteq 2^{\mathbb{N}}$ are exactly those for which $P = [T_P]$. A Π^0_1 class is a closed set for which some computable tree $T \supseteq T_P$ has $[T] = P$; in this case T_P is a Π^0_1 set. For any tree T, let $Ext(T)$ be the set of nodes of T which have an infinite extension in $[T]$, so if $P = [T]$, $Ext(T) = T_P$.

A partial computable functional $\Phi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is given by a computable representation $\varphi : \{0,1\}^* \to \{0,1\}^*$ such that $\sigma \prec \tau$ implies $\varphi(\sigma) \preceq \varphi(\tau)$; $\Phi(X)$ is defined when $\bigcup_n \varphi(X \upharpoonright n)$ is infinite, and in that case they are equal. Similar representations hold for functions on $\mathbb{N}^{\mathbb{N}}$.

An infinite set $C \subseteq \omega$ is called *immune* if it does not include any infinite c.e. subset, or equivalently if it has no infinite computable subset. A c.e. set which is the complement of an immune set is *simple*.

Definition 1.1. Let P be a closed subset of $2^{\mathbb{N}}$.

1. P is *immune* if T_P is immune.

2. P is *tree-immune* if T_P has no infinite computable subtree.

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It is easy to see that an immune closed set must be tree-immune, and both must be *special*; i.e., have no computable paths. In section 2 we separate all three notions. We also show that the class of separating sets $S(A,B)$ for any pair of computably inseparable sets A and B is immune and that any perfect thin Π^0_1 class is tree-immune. We define the notion of *prompt immunity* and construct an example of a Π^0_1 class of positive measure which is promptly immune.

In section 3, we consider connections between immunity and Binns' notion of *smallness* [5]. We show that every special hyperimmune Π^0_1 class is tree-immune and that every small special Π^0_1 class is immune. In section 4, we consider connections with the Medvedev degrees of difficulty [20, 23]. We show that for closed sets P and Q, the meet $P \oplus Q$ is (tree-)immune if and only if both P and Q are (tree-)immune, whereas the join $P \otimes Q$ is (tree-)immune if and only if at least one of P and Q are (tree-)immune. We show that for any Π^0_1 class P with no computable element, there is a non-immune Π^0_1 class Q with no computable element which is Medvedev reducible to P. In section 5, we show that no immune-free degree cups to any generalized separating class (in the sense of Cenzer and Hinman [10], and hence every immune-free Medvedev degree is non-cuppable.

In section 6, we show that any random closed set (in the sense of [2]) is immune. We also show that any random closed set is not small.

2 Immunity for Π^0_1 classes

We begin with two useful characterizations of immunity.

Lemma 2.1. *A closed set P is immune if and only if T_P has no infinite c.e. subtree.*

Proof. Certainly if T_P has an infinite c.e. subtree, then it has an infinite computable subset. For the converse, let $S \subseteq T_P$ be an infinite computable subset and define the tree T by

$$\sigma \in T \iff (\exists \tau \in S)\sigma \preceq \tau.$$

Then T is an infinite c.e. subtree of T_P. $\square$

Theorem 2.3 shows we cannot ensure that every infinite c.e. tree has an infinite computable subtree.

Lemma 2.2. *P is not immune if and only if there is a computable sequence $\{\sigma_n : n \in \omega\}$ such that $\sigma_n \in T_P \cap \{0,1\}^n$ for each n.*

Proof. The reverse implication is immediate. Now suppose that C is an infinite computable subset of T_P and enumerate C as $\{\tau_0, \tau_1, \dots\}$. Observe that C must have arbitrarily long elements and define σ_n to be $\tau_i \upharpoonright n$, where i is the least such that $|\tau_i| \geq n$. $\square$

It is clear any immune class is tree-immune, and any tree-immune class is special. The following results show that neither implication reverses.

Theorem 2.3. *There exists a tree-immune Π^0_1 class P that is not immune.*

Proof. Let S_e be the e^{th} computable tree, with characteristic function $\varphi_e : \{0,1\}^* \to \{0,1\}$. We will say that S_e has height $\geq m$ at stage s if $\varphi_{e,s}(\sigma)$ is defined for all $\sigma \in \{0,1\}^m$ and S_e has at least one node of length m.

We will build a sequence of nested computable trees T_s such that $T_P = \bigcap_s T_s$ and a prefix-free infinite c.e. set A such that $A_s = \{\sigma_0, \ldots, \sigma_s\} \subseteq Ext(T_s)$ and $|\sigma_s| > s$. We have the following requirements:

$$N_e : |S_e| = \infty \;\Rightarrow\; S_e \nsubseteq T_P.$$

Each N_e has an associated $m_s(e)$, the minimum height of S_e required before we act for N_e. For all e, $m_0(e) = 2e+1$.

To meet a single requirement N_0 we wait until the stage s when S_0 attains height ≥ 1 $(= m_0(0))$. Then we choose the leftmost τ in $S_0 \cap \{0,1\}$, let $m_s(0) = 1 + \max\{|\sigma_i| : i < s\}$, and choose all σ_t, $t \geq s$, to be incompatible with τ. Then at stage $t > s$ when S_0 reaches height $\geq m_s(0)$, we choose $\tau' \in S_e \cap \{0,1\}^{m_s(0)}$ extending τ and let T_{t+1} be the result of removing from T_t all extensions of τ'. If S_e has no extensions of τ of length $m_s(0)$, then if $\tau = 1$, or $\tau = 0$ but $1 \notin S_e$, we abandon N_0, as S_e is finite. Otherwise we reset τ to 1 and $m_t(0) = 1+\max\{|\sigma_i| : i < t\}$ and wait again, avoiding the cone above the new τ (and no longer avoiding the old) in future σ_i choices.

The same module holds for all other requirements; we maintain a set R of bases of cones that must be avoided by A. Each $m_s(e)$ changes its value at most 2^{2e+1} times, and the values it takes on are sufficiently large that standard measure arguments show we always have room to choose new σ_i nodes and maintain their extendibility.

Stage 0: $\forall e\; m_0(e) = 2e+1$; $A_0 = R_0 = \emptyset$; $T_0 = \{0,1\}^*$.

Stage $s > 0$: Step 1. For each $e \leq s$ such that S_e has height $\geq 2e+1$ newly at stage s, set $m_s(e) = 2e+1+\max\{|\sigma_i| : i < s\}$ and set τ_e to the leftmost string in $S_e \cap \{0,1\}^{e+1}$. Enumerate all such τ_e into R_s.

Step 2. For each $e \leq s$ such that $m_{s-1}(e) > 2e+1$, S_e has height $m_{s-1}(e)$ newly at s, and $S_e \cap \{0,1\}^{m_{s-1}(e)} \subseteq T_{s-1}$, if there exists a string $\tau \succ \tau_e$ in $S_e \cap \{0,1\}^{m_{s-1}(e)}$ remove the leftmost such from T_s. If there does not exist such a τ, remove τ_e from R_s. If τ_e is the rightmost string in $S_s \cap \{0,1\}^{2e+1}$, do nothing. Otherwise choose the leftmost of the strings to the right of τ_e, label it the new τ_e, put this new τ_e into R_s, and set $m_s(e) = 2e+1+\max\{|\sigma_i| : i < s\}$.

Step 3. For any e not treated above, let $m_s(e) = m_{s-1}(e)$; let T_s be T_{s-1} minus the strings removed in the previous step (if any) and all their extensions.

Step 4. Finally, let Q be the part of T_s uncovered by A and R. That is,

$$Q = T_s - \{\tau\widehat{\ }\rho : \tau \in A_{s-1} \cup R_s, \rho \in \{0,1\}^*\}.$$

Note that since we only remove strings from T that are within the intervals of permanent members of R, we would get the same Q if we replaced T_s with $\{0,1\}^*$. Choose the leftmost $\sigma \in Q$ of length at least $s+2$ and let it be $\sigma_s \in A_s$.

To verify the construction works, first note every σ_i has an extension by a straightforward measure argument: we remove at most one node τ on behalf of each S_e, and for any i such that $\tau \succeq \sigma_i$, we ensure $\mu(I(\tau)) \leq 2^{-2e-1-|\sigma_i|}$. The sum of the measure removed from any $I(\sigma_i)$ is hence bounded by $\frac{2}{3}\mu([\sigma_i])$.

Another measure argument shows there is always enough room in Q to choose a new string in A without covering all of T_s. Since each S_e has at most one node in R at a time, the measure of Q at stage s is at least

$$x = 1 - \sum_{e=0}^{s} 2^{-2e-1} - \sum_{i=1}^{s-1} 2^{-i-2},$$

which we need to be greater than (at most) 2^{-s-2}. It is easily checked that $x - 2^{-s-2}$ is

$$\frac{1}{12} + \frac{1}{3 \cdot 2^{2s+1}} + \frac{1}{2^{s+2}},$$

which is clearly positive.

Since it is clear that the requirements are met, P is a Π^0_1 class, and $A \subset T_P$ is computable, the proof is complete. $\square$

Theorem 2.4. *There is a special Π^0_1 class that is not tree-immune.*

Proof. This is a corollary of Theorem 4.8; any Q^* where Q is special is also special but not tree-immune. $\square$

The next results show many Π^0_1 classes of interest are immune. Recall $S(A,B)$ denotes the class of separating sets for A and B (all C such that $A \subseteq C$ and $B \cap C = \emptyset$); it is a closed set, and when A and B are c.e. it is a Π^0_1 class.

Proposition 2.5. *If A and B are computably inseparable, then $S(A,B)$ is immune.*

Proof. Suppose that $W \subset T_{S(A,B)}$ is an infinite c.e. set, enumerated without repetition as $\sigma_0, \sigma_1, \ldots$. Note that for any $\sigma \in W$ and any $n < |\sigma|$, $n \in A \Rightarrow \sigma(n) = 1$ and $n \in B \Rightarrow \sigma(n) = 0$. Since W must have elements of arbitrary length, we may computably define $i(n)$ to be the least i such that $|\sigma_i| > n$, and let $X(n) = \sigma_{i(n)}(n)$ to compute a separating set for A and B. $\square$

The notion of a *thin* Π^0_1 class corresponds to that of a hyperhyperimmune set and has been studied extensively by many researchers in articles including [8, 11, 15]. A Π^0_1 class P is thin if for any Π^0_1 class $Q \subset P$, there is a clopen set U such that $Q = P \cap U$. This is equivalent to saying that the family of Π^0_1 subsets of P is complemented, that is, for any Π^0_1 class $Q \subset P$, $P \setminus Q$ is also a Π^0_1 class. Since any hyperhyperimmune set is also immune, the following result is natural.

Proposition 2.6. *If P is a perfect thin Π^0_1 class, then P is tree-immune.*

Proof. Let P be perfect thin (and therefore having no computable member) and suppose that some infinite computable tree $W \subseteq T_P$. Let L be the set of *leaves* of W, that is

$$L = \{\sigma \in W : \sigma^\frown 0 \notin W \ \& \ \sigma^\frown 1 \notin W\}.$$

Then the elements of L are pairwise incomparable and, since P has no computable elements, L is infinite. To see this, note that if L were finite, then $Ext(W)$ would be computable and thus W would have a computable element (in particular the leftmost path), which would also belong to P. That is, suppose that L were finite and let m be the maximum length of a node in L, then, for any σ,

$$\sigma \in Ext(W) \iff (\exists \tau \in W \cap \{0,1\}^{m+1})\sigma \prec \tau.$$

Note that for each $\sigma \in L$, $\sigma \in T_P$. Now we can partition P into the following subsets:

$$P_0 = \{X \in P : (\forall n) X\lceil n \notin L\}$$

and

$$P_1 = \{X \in P : (\exists n) X\lceil n \in L\}.$$

P_0 is a Π^0_1 class and therefore, since P is thin, P_1 is also a Π^0_1 class.

Let $L = \{\sigma_0, \sigma_1, \dots\}$ and observe that the closed set P_1 is covered by the family $\{I(\sigma_i) : i \in \omega\}$. It follows by compactness that $P_1 \subseteq I(\sigma_0) \cup \cdots \cup I(\sigma_k)$ for some finite k. But this contradicts the fact that every $\sigma_i \in T_P$ and that the σ_is are pairwise incomparable. $\square$

A c.e. set A is called *promptly simple* if for some enumeration $\{A_n\}_{n\in\mathbb{N}}$ of A there is a computable function π such that for any infinite c.e. set $W_e \subseteq \mathbb{N}$ there are n, s with $n \in W_{e,s+1} - W_{e,s}$ and $n \in A_{\pi(s)}$.

For P a Π^0_1 class, let T be a computable tree giving P. For each s, let T_s be the collection of nodes of T which have length-s extensions in T. Let $\{\sigma_n\}_{n\in\mathbb{N}} = \{\lambda, 0, 1, 00, 01, 10 \dots\}$ denote the length-lexicographical ordering of the elements of $\{0,1\}^*$. We say that P is *promptly immune* if there is a computable function π such that for any infinite c.e. set W, there exist n, s such that

$$n \in W_{s+1} - W_s \ \ \& \ \ \sigma_n \notin T_{\pi(s)}.$$

There exist Π^0_1 classes with positive measure which have no computable elements. The next result is an improvement on this.

Theorem 2.7. *There exists a Π^0_1 class P of positive measure which is promptly immune.*

Proof. We define the Π^0_1 class $P = [T]$ in stages T_s and let $T = \bigcap_s T_s$. P will be promptly immune via the function $\pi(s) = s+1$. For each e, we will wait for some n such that $|\sigma_n| > 2e$ to come into W_e at stage $s+1$ and then remove σ_n from T_{s+1} by removing σ_n and all extensions (if any) from T. Initially $T_0 = \{0,1\}^*$. After stage s, we will have satisfied some of the requirements. At stage $s+1$, we look for the least $e \leq s$ which has not yet been satisfied and such that some suitable $n \in W_{e,s+1} - W_{e,s}$. We meet this requirement by setting $T_{s+1} = T_s - \{\tau : \sigma_n \preceq \tau\}$. Note that this action removes from $[T]$ a set of measure $\leq 2^{-2e-1}$, so that the total measure removed is

$$\leq \sum_e 2^{-2e-1} = \frac{2}{3}.$$

It follows that $T_s \neq \emptyset$ for any s and therefore $P = [T]$ is not empty, and in fact has measure at least $\frac{1}{3}$. □

3 Smallness and Hyperimmunity

In this section, we compare immunity with other "smallness" notions for Π^0_1 classes. Some definitions are needed.

There is a one-to-one correspondence between the set of natural numbers and the set of finite subsets of natural numbers, given as follows. For any $n > 0$, let n be uniquely expressed in binary form as $n = \sum_{j=1}^k 2^{e_j}$ for some finite sequence $e_1 < e_2 < \cdots < e_k$; the finite set $\{e_1, \ldots, e_k\}$ is denoted by D_n and n is its *canonical index*. We set $D_0 = \emptyset$. For any computable function f, the sequence $D_{f(n)}$ is called a *strong array*; it is called *disjoint* if the sets $D_{f(n)}$ are pairwise disjoint.

A set $C \subseteq \mathbb{N}$ is called *hyperimmune* if there is no disjoint strong array $\langle D_{f(n)} \rangle$ such that, for all n, $D_{f(n)} \cap C \neq \emptyset$. A well-known theorem by Kuznecov, Medvedev, and Uspenski ([25] V.2.3) states that $C = \{c_0 < c_1 < \ldots\}$ is hyperimmune if and only if there is no infinite computable function g such that $g(n) > c_n$ for all n.

A finite string $\sigma \in \{0,1\}^n$ has Gödel number $\sum_{i=0}^{n} \sigma(i)2^i$. If F is a finite set of (Gödel numbers of) strings, then $F^* = \bigcup\{I(\sigma) : \sigma \in F\}$. Binns [6] called a sequence $\langle D_{f(n)} \rangle$ of finite sets of (Gödel numbers of) strings a *disjoint strong array* if the sets $D^*_{f(n)}$ are pairwise disjoint.

Definition 3.1.

1. (Binns [5]). A closed set P is *small* if there is no computable function g such that, for all n, $card(\{0,1\}^{g(n)} \cap T_P) > n$.
2. (Binns [6]). A closed set P is *hyperimmune* if there is no disjoint strong array $\langle D_{f(n)} \rangle$ such that $P \cap D^*_{f(n)} \neq \emptyset$ for all n.

Binns [6] showed that the class DNC_2 of diagonally non-computable functions is not small, and in fact not hyperimmune. By Proposition 2.5, this gives an example of an immune class of measure 0 which is not small. It is also easy to see that a class of positive measure cannot be small, so the immune class of Theorem 2.7 is also not small.

For any tree $T \subseteq \{0,1\}^*$, we say that σ is a *branching node* of T if both $\sigma^\frown 0$ and $\sigma^\frown 1$ are in T; let $Br(T)$ denote the set of branching nodes of T.

Theorem 3.2 (Binns [5]). *A Π^0_1 class P is small if and only if $Br(T_P)$ is hyperimmune.*

Theorem 3.3 (Binns [6]). *Every small Π^0_1 class is hyperimmune.*

The converse to Theorem 3.3 does not hold. It is not clear whether every special hyperimmune Π^0_1 class must be immune, because the nodes witnessing immunity need not be incomparable. However, we have the following result.

Theorem 3.4. *Every special hyperimmune Π^0_1 class is tree-immune.*

Proof. Assume P is not tree-immune, and let $T \subseteq T_P$ be a computable tree. Since P is special, T has an infinite, computable set $L = \{\sigma_0, \sigma_1, \dots\}$ of leaves. Then we may define a disjoint strong array

$$D_{f(n)} = \{\sigma_n\}.$$

Hence P is not hyperimmune. □

Cenzer, Weber, and Wu [14] asked whether every small special Π^0_1 class is immune. We can now answer this question.

Theorem 3.5. *Every small special Π^0_1 class is immune.*

Proof. Suppose that P is special and small but not immune, and let $T \subseteq T_P$ be an infinite c.e. subtree.

Claim 3.6. *$Br(T)$ is infinite.*

Proof. Suppose by way of contradiction that $Br(T)$ is finite and let s be the maximum length of any $\sigma \in Br(T)$. It follows that any node in T of length $\geq s$ must extend one of the finite set of nodes of length s. Since T is infinite, there must be a single node $\tau \in T$ which has infinitely many extensions in T. Since there is no branching above τ, all of those extensions are comparable, so that there is a unique infinite path X in T through τ. Since T is c.e., we may compute the path X as follows. Given i, enumerate the elements of T until we find a string σ with $|\sigma| \geq i$ which is comparable with τ and then $X(i) = \sigma(i)$. This violates the assumption that P is special. $\square$

Now $Br(T)$ is itself a c.e. set, since we can enumerate $\sigma \in Br(T)$ once σ, $\sigma^\frown 0$, and $\sigma^\frown 1$ have all been enumerated into T. Hence $Br(T)$ has an infinite, increasing computable subset and is certainly not hyperimmune. It follows that the larger set $Br(T_P)$ is also not hyperimmune, so by Theorem 3.2 P is not small. $\square$

4 Degrees of Difficulty

Π^0_1 classes are often viewed as collections of solutions to some mathematical problem. Muchnik and Medvedev reducibility, defined for closed subsets of $2^{\mathbb{N}}$ and indeed $\mathbb{N}^{\mathbb{N}}$ in general, order classes based on this viewpoint. The class A is *Muchnik* (a.k.a. *weakly*) *reducible* to the class B ($A \leq_w B$) if for every $X \in B$ there is $Y \in A$ such that $Y \leq_T X$ [21]. The class A is *Medvedev* (a.k.a. *strongly*) *reducible* to B ($A \leq_s B$) if there is a single Turing reduction procedure which, when given any element of B as an oracle, computes an element of A; it is exactly the uniformization of Muchnik reduction [20]. These reductions have been studied extensively by Binns (e.g., [4]), Cenzer and Hinman [9, 10] and Simpson (e.g., [24]) and have connections to randomness [22]. We will need the result from [9] that any partial computable $\Phi : P \to Q$ for two Π^0_1 classes P and Q may be extended to a total computable functional. The Medvedev *degrees* are equivalence classes under $P \equiv_s Q$, defined as ($P \leq_s Q$) & ($Q \leq_s P$), and similarly for the Muchnik degrees. Let $\mathcal{P}_s$ denote the partial ordering of the Medvedev degrees of Π^0_1 classes.

Proposition 4.1. *If P is not (tree-)immune and Q is Medvedev reducible to P then Q is also not (tree-)immune.*

Proof. Let P be a Π^0_1 class which is not tree-immune, and $V \subseteq T_P$ an infinite computable tree. Let Φ witness $Q \leq_s P$ and set $S = \Phi(V)$; note that S is a tree. By the definition of partial computable functional and the fact that Φ must be defined on all of P, $S \subseteq T_Q$ and S is infinite. It remains to show S is computable.

To determine whether $\tau \in S$, compute $\varphi(\sigma)$ for all $\sigma \in 2^{<\omega}$ in lexicographical order until $|\varphi(\sigma)| \geq |\tau|$ for all $\sigma \in P$ of some length n. Then $\tau \in S$ if and only if $\tau \preceq \varphi(\sigma)$ for some $\sigma \in V \cap \{0,1\}^n$.

If P is not immune, then there is an infinite c.e. tree $V \subseteq T_P$ and the argument above shows that $\Phi(V)$ is an infinite c.e. subtree of T_Q, so that Q is also not immune. □

Let us say that a Medvedev degree $\mathbf{d} \in \mathcal{P}_s$ is *(tree-)immune* if there is some class $P \in \mathbf{d}$ which is (tree-)immune and otherwise $\mathbf{d}$ is *(tree-)immune-free.*

Corollary 4.2. *1. If $\mathbf{d} \in \mathcal{P}_s$ contains a non-(tree-)immune Π^0_1 class, then $\mathbf{d}$ is (tree-)immune-free.*

2. If $\mathbf{d} \in \mathcal{P}_s$ contains a (tree-)immune Π^0_1 class, then every member of $\mathbf{d}$ is (tree-)immune.

For $X, Y \in 2^{\mathbb{N}}$, the join $X \oplus Y = Z$ is given by $Z(2n) = X(n)$ and $Z(2n+1) = Y(n)$. Similarly, for finite sequences σ and τ of equal length, we may define $\sigma \oplus \tau = \rho$, where $\rho(2n) = \sigma(n)$ and $\rho(2n+1) = \tau(n)$.

The quotient structure of the Π^0_1 classes under either Muchnik or Medvedev equivalence is a lattice, and both have the same join and meet operators. The join of P and Q is given by

$$P \otimes Q = \{X \oplus Y : X \in P, Y \in Q\}.$$

If $P = [S]$ and $Q = [T]$, then $P \otimes Q = [S \otimes T]$, where

$$S \otimes T = \{\sigma \oplus \tau, (\sigma \oplus \tau)i : \sigma \in S, \tau \in T, |\sigma| = |\tau|, i \in \{0,1\}\};$$

since all finite joins are of even length, we branch at odd levels. The meet of P and Q is given by

$$P \oplus Q = \{0^\frown X : X \in P\} \cup \{1^\frown Y : Y \in Q\}.$$

If $P = [S]$ and $Q = [T]$, then $P \oplus Q = [S \oplus T]$, where

$$S \oplus T = \{0^\frown \sigma : \sigma \in S\} \cup \{1^\frown \tau : \tau \in T\}.$$

Binns [6] showed that $P \oplus Q$ and $P \otimes Q$ are small if and only if both P and Q are small. The results for immunity are not quite the same.

Theorem 4.3. *For any closed sets P and Q, $P \oplus Q$ is (tree-)immune if and only if both P and Q are (tree-)immune.*

Proof. Suppose first that P is not immune and let $C \subseteq T_P$ be an infinite computable set. Then $\{0^\frown\sigma : \sigma \in C\}$ is a computable subset of $T_{P\oplus Q}$. Supposing P is not tree-immune, let $V \subseteq T_P$ be an infinite computable tree. Then $\{\lambda\} \cup \{0^\frown\sigma : \sigma \in V\}$ is an infinite computable subtree of $T_{P\oplus Q}$. The arguments when Q is not (tree-)immune are, of course, symmetric.

Next suppose that $P \oplus Q$ is not immune and let $C \subseteq T_{P\oplus Q}$ be an infinite computable set. Let $C_i = \{\sigma : i^\frown\sigma \in C\}$ for $i = 0, 1$. Then $C_0 \subseteq T_P$, $C_1 \subseteq T_Q$ and both sets are computable. Clearly either C_0 is infinite or C_1 is infinite, which implies that either P is not immune or Q is not immune. A similar argument applies if $P \oplus Q$ is not tree-immune, where $V \subseteq T_{P\oplus Q}$ is an infinite computable tree and the corresponding $V_0 \subseteq T_P$ and $V_1 \subseteq T_Q$ are computable trees. $\square$

Theorem 4.4. *For any closed sets P and Q, $P \otimes Q$ is (tree-)immune if and only if at least one of P and Q is (tree-)immune.*

Proof. Suppose first that $P \otimes Q$ is not tree-immune and let $V \subseteq T_{P\otimes Q}$ be an infinite computable tree. Let

$$V_P = \{\sigma : (\exists \tau \in \{0,1\}^{|\sigma|})(\sigma \oplus \tau \in V)\}$$

and similarly

$$V_Q = \{\tau : (\exists \sigma \in \{0,1\}^{|\tau|})(\sigma \oplus \tau \in V)\}.$$

Then V_P is an infinite computable subtree of T_P and V_Q is an infinite computable subtree of T_Q, so that neither P nor Q is tree-immune. A similar argument applies if $P \otimes Q$ is not immune, where V, V_P and V_Q are now infinite c.e. trees.

Next suppose that both P and Q are not tree-immune and let $V_P \subseteq T_P$ and $V_Q \subseteq Q$ be infinite computable trees. Then $V_P \otimes V_Q$ is an infinite computable subtree of $T_P \otimes T_Q = T_{P\oplus Q}$. A similar argument applies if P are Q are both not immune, where V_P, V_Q and $V_P \otimes V_Q$ are all infinite c.e. trees. $\square$

Corollary 4.5. *The immune-free degrees and the tree-immune-free degrees each form a prime ideal in the lattice $\mathcal{P}_s$.*

Corollary 4.6. *The tree-immune-free Medvedev degrees form a proper subideal of the immune-free Medvedev degrees.*

Proof. Let $\mathbf{d}$ be the Medvedev degree of the tree-immune, non-immune Π^0_1 class P constructed in Theorem 2.3. Then by Corollary 4.2, $\mathbf{d}$ is tree-immune but immune-free. $\square$

We now turn to questions of density. Let 0_s denote the least Medvedev degree, which consists of all Π^0_1 classes that have a computable member. Binns has shown there is a nonsmall class of every nonzero Medvedev degree. We have the following bounding result for nonimmune classes.

Theorem 4.7. *For any nonzero Π^0_1 class P, there is a Π^0_1 class Q with $0_s <_s Q \leq_s P$ which is not tree-immune, and hence not immune.*

Proof. Let R be the Π^0_1 class of Theorem 2.4 which is nonzero and not tree-immune. It follows from Theorem 4.3 that $P \oplus R$ is not tree-immune, but it is also special and certainly $P \oplus R \leq_s P$. $\square$

Theorem 4.8. *For every Π^0_1 class Q, there exists a Π^0_1 class $Q^* \leq_s Q$ such that Q^* has tree-immune-free Medvedev degree, and Q^* is Muchnik equivalent to Q. Furthermore, if Q is immune, then $Q^* <_s Q$.*

Proof. The case that Q is not special is obvious. Let Q be a special Π^0_1 class, and let T be a computable tree such that $[T] = Q$. We note that the set L of all leaves of T is computable. We set

$$T^* = T \cup \{\sigma^\frown\tau : \sigma \in L \ \&\ \tau \in T\}.$$

and let $Q^* = [T^*]$, so that

$$Q^* = Q \cup \{\sigma^\frown X : \sigma \in L \ \&\ X \in Q\}.$$

Then Q^* is a Π^0_1 class and $Q \subseteq Q^*$, so $Q^* \leq_s Q$. T is a computable subtree of T_{Q^*}, so that Q^* is not tree-immune, and hence by Corollary 4.2, Q^* has tree-immune-free degree. At the same time, every member of Q^* is Turing equivalent to a member of Q, so that Q^* is Muchnik equivalent to Q.

If Q is immune, it follows from Proposition 4.1 that we may not have $Q \leq_s Q^*$, since Q^* is not immune. $\square$

Lemma 4.9 (Essentially by Simpson [23]). *There exists a Medvedev complete set Q and a computable function q such that, for any e, the e^{th} Π^0_1 class P_e is Medvedev reducible to Q via a computable functional $\Phi_{q(e)}$.*

Remark 4.10. Every Medvedev complete set has this property.

Lemma 4.11. *Let $P \leq_s Q$ be special Π^0_1 classes, S and T computable trees with $[S] = P$ and $[T] = Q$, and L_S and L_T the computable sets of all leaves of S and T, respectively. Then there is a computable functional Φ^* such that $\Phi^*(Q) \subseteq P$ and $\Phi^*(L_T) \subseteq L_S$.*

Proof. Since P is special, any $\sigma \in S$ has an extension in L_S. Assume $P \leq_s Q$ via the computable functional Φ and let φ be a representing function for Φ. We construct the desired functional Φ^* with representing function φ^*.

First suppose $\tau \in 2^{<\omega}$ has no initial segment which is a leaf of T. If $\varphi(\tau) \in S$, then we let $\varphi^*(\tau) = \varphi(\tau)$. If $\varphi(\tau) \notin S$, then we let σ be the longest initial segment of $\varphi(\tau)$ which belongs to S, so that $\sigma \in L_S$, and let $\varphi^*(\tau) = \sigma$. Note that if $X \in Q$, it follows that $\varphi^*(X \upharpoonright n) = \varphi(X \upharpoonright n)$, so that $\Phi^*(Q) \subseteq P$ as desired.

Next suppose that $\tau \succeq \sigma$ for some leaf σ of T. If $\varphi(\sigma) \notin S$, then as above let $\varphi^*(\sigma)$ be the longest initial segment of $\varphi(\sigma)$ which belongs to S. If $\varphi(\sigma) \in S$, let $\varphi^*(\sigma)$ be the shortest and leftmost leaf of S which extends $\varphi(\sigma)$. Then let $\varphi^*(\tau) = \varphi^*(\sigma)^\frown 0^{|\tau|-|\sigma|}$. It follows that φ^* maps L_T into L_S.

It is easy to check that φ^* is monotonic and defines a computable functional Φ^*. $\square$

Theorem 4.12. *There is a greatest tree-immune-free Medvedev degree.*

Proof. Let Q be a Medvedev complete set, T a computable tree such that $Q = [T]$ and Q^* as defined in Theorem 4.8. Fix any non-tree-immune Π^0_1 class P and let $V \subseteq T_P$ be an infinite computable tree. We may assume that P has no computable path. By the Medvedev completeness of Q and Lemma 4.11, $[V] \leq_s Q$ via some computable functional Φ with representing function φ such that φ maps L_T into L_V.

Let f be a computable function such that $P_{f(\sigma)} = P \cap I(\sigma)$ for all $\sigma \in L_V$, and observe that since $V \subseteq T_P$, $P_{f(\sigma)}$ is a nonempty subset of P.

We now construct a computable functional $\Psi : Q^* \to P$. Let $X \in Q^*$. We define the partial output $\psi(X \upharpoonright n)$ as follows. As long as $\varphi(X \upharpoonright n) \in V$, simply let $\psi(X \upharpoonright n) = \varphi(X \upharpoonright n)$. If $\varphi(X \upharpoonright n) \in V$, but $\varphi(X \upharpoonright n+1) \notin V$, then there exists $\sigma \in L_V$ with $\varphi(X \upharpoonright n) \preceq \sigma \prec \varphi(X \upharpoonright n+1)$. Furthermore, since $\Phi : Q \to [V]$ and $\varphi(X \upharpoonright n+1) \notin V$, it follows that $X \notin Q$. In this case, it follows by the assumption from Lemma 4.11 that $X \upharpoonright n+1 \in L_V$. To see this, let k be the least such that $X \upharpoonright k \in L_V$. Then $\varphi(X \upharpoonright k) = \sigma$ by the assumption from Lemma 4.11 and the monotonicity of φ. Also $k \leq n$ since $\varphi(X \upharpoonright n+1) \notin V$ and hence $\varphi(X \upharpoonright n) = \sigma$ as well.

Now define $\Psi(X) = \Phi_{q(f(\sigma))}(X)$, where q is the function from Lemma 4.9. Since we know $\sigma \prec \Phi_{q(f(\sigma))}(X)$, we can let $\psi(X \upharpoonright n+r) = \sigma \cup \varphi_{q(f(\sigma))}(X \upharpoonright n+r)$, that is, $\psi(X \upharpoonright n+r) = \sigma$ if $\varphi_{q(f(\sigma))}(X \upharpoonright n+r) \preceq \sigma$ and otherwise $\psi(X \upharpoonright n+r) = \varphi_{q(f(\sigma))}(X \upharpoonright n+r)$. $\square$

Corollary 4.13. *The c-immune-free Medvedev degrees forms a principal prime ideal in $\mathcal{P}_s$.*

5 Non-Cupping

Cenzer-Weber-Wu [14] suggested the problem of determining the cuppable Π^0_1 classes in $\mathcal{P}_s$. Here we say that an incomplete Π^0_1 class P is *cuppable* if there exists an incomplete Π^0_1 class Q such that $P \otimes Q$ is Medvedev complete. In general, P *cups* to $R >_s P$ if there exists $Q <_s R$ such that $P \otimes Q \equiv_s R$.

The first result in this direction is the following.

Theorem 5.1 (Simpson [23]). *Any Π^0_1 class that cups to a separating class must have measure 0.*

Hence, the positive measure Medvedev degrees `POS` form a subideal of Medvedev non-cupping degrees `NCup`, and, by Theorem 2.7, a non-cuppable promptly immune Π^0_1 class exists. However, we will observe a further relationship between immunity and non-cuppability.

Recall for disjoint sets A, B, $S(A, B)$ is the class of all separating sets $C \supseteq A$, $C \cap B = \emptyset$. In particular, $DNC_2 = S(A_0, A_1)$ where $A_i = \{e : \varphi_e(e) = i\}$. A *generalized separating class* is the product $\prod_n F_n$ where $\{F_n\}_{n\in\omega}$ is a computable sequence of finite subsets of $\mathbb{N}$. For $S(A, B)$ the set $F_n = \{0\}$ if $n \in B$, $\{1\}$ if $n \in A$ and $\{0, 1\}$ if $n \notin A \cup B$. Generalized separating classes were studied by Cenzer and Hinman [10]. It is important to note that any generalized separating class P is *computably bounded* and hence is computably homeomorphic to a Π^0_1 class $Q \subseteq \{0, 1\}^\omega$ (see Lemma 1.3 of [7]). Hence the Medvedev degrees of the generalized separating classes are included in the Medvedev degrees of subsets of Cantor space.

Theorem 5.2. *No immune-free degree cups to any generalized separating class.*

Proof. Let P be an non-immune Π^0_1 class and $V \subseteq T_P$ an infinite computable set, with fixed enumeration $\{\sigma_i\}_{i\in\omega}$. Let $S = \prod_n F_n$ be a generalized separating class for a sequence $\{F_n\}_{n\in\omega}$ of finite sets. Suppose that for some Q, $S \le P \otimes Q$ via a computable functional Φ. We will write an input $X \oplus Y$ to Φ as the ordered pair X, Y.

We construct a computable functional Ψ witnessing $S \le_s Q$. Given $Y \in Q$, define $Z = \Psi(Y)$ as follows. For each n, let $Z(n) = \Phi(\sigma_i, Y)(n)$, where i is the least such that $\Phi(\sigma_i, Y)(n)$ is defined. We know that such i exists since, by compactness, there is some m such that $|\varphi(\sigma, \tau)| > n$ for all $\sigma \in T_P$, $\tau \in T_Q$ with length $\ge m$.

It remains to confirm that $Z = \Psi(Y) \in S$; that is, $Z(n) \in F_n$ for all n. Given n and $\sigma_i \in V$ such that $Z(n) = \Phi(\sigma_i, Y)(n)$, we can find $X \in P$ such that $\sigma_i \prec X$ (since $\sigma_i \in T_P$). It follows that $\Phi(X, Y) \in S$ and hence $\Phi(\sigma_i, Y)(n) = \Phi(X, Y)(n) \in F_n$. □

Corollary 5.3. *Every immune-free Medvedev degree is Medvedev non-cuppable.*

Proof. The class of 2-valued diagonally noncomputable functions, DNC_2, is a Medvedev complete generalized separating class, and hence no immune-free degree can cup to it. □

We get new subideals $\overline{\mathtt{IM}}$ and $\overline{\mathtt{TIM}}$ of Medvedev non-cuppable degrees $\mathtt{NCup}$, which consist of immune-free and tree-immune-free degrees, respectively. However, immunity does not necessarily give a cupping property. Actually, as seen before, a positive measure promptly immune degree in Cenzer-Weber-Wu [14] is an example of a non-cuppable immune degree.

Corollary 5.4. *A Muchnik complete Medvedev non-cuppable degree exists.*

Proof. By Theorem 4.12, $\max \overline{\mathtt{TIM}}$ exists and it is clearly Muchnik complete since it is degree-isomorphic to any Medvedev complete class. Moreover, it is non-cuppable by Corollary 5.3. □

Theorem 5.5. *For* $\mathbf{c} = \max \overline{\mathtt{TIM}}$*, a measure* 0 *immune non-cuppable degree* $> \mathbf{c}$ *exists.*

Proof. Let $\mathbf{d}$ be a positive measure, promptly immune Medvedev degree. Then $\mathbf{d} \not\leq \mathbf{c}$ holds since tree-immune-free degrees are downward closed. We claim $\mathbf{a} = \mathbf{c} \cup \mathbf{d}$ is the desired degree. This follows from the results that immune degrees and non-cuppable degrees form ideals, positive measure-free degrees form a filter, and $\mathbf{d}$ has positive measure-free degree by its Muchnik completeness (see Simpson [23]). □

Corollary 5.6. *The immune-free Medvedev degrees* $\overline{\mathtt{IM}}$ *and the tree-immune-free Medvedev degrees* $\overline{\mathtt{TIM}}$ *form proper subideals of the noncuppable Medvedev degrees* $\mathtt{NCup}$.

6 Immunity and randomness

Finally we consider the immunity of random closed sets. A closed set P may be coded as an element of $3^{\mathbb{N}}$; P is called random if that sequence is Martin-Löf random (for background on randomness see [16]). The code of P is defined from T_P; the nodes of T_P are considered in order by length and then lexicographically, and each one is represented in the code by 0, 1, or 2 according to whether the node has only the left child, only the right child, or both children, respectively. Randomness for closed sets is defined and explored in [2, 3], where it is shown among other results that no Π^0_1 class is random, and that no random closed set contains an f-c.e. path for any computable f bounded by a polynomial. The following theorem does not follow immediately but is not surprising.

Theorem 6.1. *If* P *is a random closed set, then* P *is immune.*

Proof. Fix a computable sequence $C = (\sigma_1, \sigma_2, \dots)$ such that $|\sigma_n| = n$ for each n. For $n > 0$, let $S_n = \{Q : (\forall i \leq n)\ \sigma_i \in T_Q\}$. Then S_n is a clopen set in the space of closed sets and the sequence $\{S_n : n \in \omega\}$ is uniformly c.e. It is clear that $C \subseteq T_P$ if and only if $P \in S_n$ for all n. Now consider the Lebesgue measure $\mu(S_n)$. Certainly $\mu(S_1) = 2/3$. Given $\mu_n = \mu(S_n)$ and σ_{n+1}, let $i \leq n$ be the largest such that $\sigma_i \prec \sigma_{n+1}$. Then $\mu_{n+1} = (\frac{2}{3})^{n+1-i}\mu_n \leq \frac{2}{3}\mu_n$. Hence $\mu(S_n) \leq (\frac{2}{3})^n$ for each n. It follows that $\{S_{2n} : n \in \omega\}$ is a Martin-Löf test and hence no random closed set can belong to every S_n. Hence if P is random, C is not a subset of T_P. Since this holds for every such C, it follows that random closed sets are immune. □

Since a random ternary sequence must contain $\frac{1}{3}$ 2s in the limit, intuitively the tree it codes must branch too much to be small. This is a straightforward consequence of the following, which is drawn from Lemma 4.5 in [2].

Lemma 6.2. *Let Q be a random closed set. Then there exist a constant $C \in \mathbb{N}$ and $k \in \mathbb{N}$ such that for all $m > k$,*

$$C\left(\frac{4}{3}\right)^m \left(1 - m^{-\frac{1}{4}}\right) \; < \; card(T_Q \cap \{0,1\}^m) \; < \; C\left(\frac{4}{3}\right)^m \left(1 + m^{-\frac{1}{4}}\right).$$

Corollary 6.3. *If Q is a random closed set, Q is not small.*

Proof. For C, k as in Lemma 6.2, define the function $g(n)$ as

$$g(n) = \max\left\{k+1, \min\left\{m : n < C\left(\frac{4}{3}\right)^m \left(1 - m^{-\frac{1}{4}}\right)\right\}\right\}.$$

It is clear that g is computable, and by Lemma 6.2, for all n the number of branches at level $g(n)$ will be at least n. □

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