Immunity and Non-Cupping for Closed Sets

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Abstract

We extend the notion of immunity to closed sets and to \( \Pi_1^0 \) classes in particular in two ways: immunity meaning the corresponding tree has no infinite computable subset, and tree-immunity meaning it has no infinite computable subtree. We separate these notions from each other and that of being special, and show separating classes for computably inseparable c.e. sets are immune and perfect thin classes are tree-immune. We define the notion of prompt immunity and construct a positive-measure promptly immune \( \Pi_1^0 \) class. We show that no immune-free \( \Pi_1^0 \) class \( P \) cups to the Medvedev complete class \( DNC \) of diagonally noncomputable sets, where \( P \) cups to \( Q \) in the Medvedev degrees of \( \Pi_1^0 \) classes if there is a class \( R \) such that the product \( P \otimes R \equiv_M Q \). We characterize the interaction between (tree-)immunity and Medvedev meet and join, showing the (tree-)immune degrees form prime ideals in the Medvedev lattice. We show that every random closed set is immune and not small, and every small special class is immune.

Keywords: Computability, \( \Pi_1^0 \) Classes
1 Introduction

The notion of a simple c.e. set and the corresponding complementary notion of an immune co-c.e. set are fundamental to the study of c.e. sets and degrees. Together with variations and related notions such as effectively immune, promptly simple, hyperimmune and so forth, they permeate the classic text of R.I. Soare [25] and its updated version.

Many of the results on c.e. sets and degrees have found counterparts in the study of effectively closed sets (Π₀¹ classes). See the surveys [12, 13] for examples. In particular, hyperhyperimmune co-c.e. sets correspond to thin Π₀¹ classes [8, 11, 15] and hyperimmune co-c.e. sets correspond to several different notions including smallness studied by Binns [5, 6].

In this paper we consider the notion of immune sets as applied to Π₀¹ classes and closed sets in general. We work in 2ᴺ with the topology generated by basic clopen sets called intervals. For any σ ∈ {0, 1}∗ the interval I(σ) is \{X : σ ⪯ X\}, where ⪯ means initial segment. Notation is standard; we note that λ denotes the empty string, σ↾n is the length-n initial segment of σ, and if T ⊆ {0, 1}∗ is a tree (i.e., it is closed under initial segment), [T] ⊆ 2ᴺ denotes the set of infinite paths through T. A node σ ∈ T is a leaf of T if σ⌢i /∈ T for any i. For any set P ⊆ 2ᴺ, we may define the tree TP = \{σ ∈ {0, 1}∗ : I(σ) ∩ P ≠ ∅\}; the closed sets P ⊆ 2ᴺ are exactly those for which P = [TP]. A Π₀¹ class is a closed set for which some computable tree T ⊇ TP has [T] = P; in this case TP is a Π₀¹ set. For any tree T, let Ext(T) be the set of nodes of T which have an infinite extension in [T], so if P = [T], Ext(T) = TP.

A partial computable functional Φ : 2ᴺ → 2ᴺ is given by a computable representation ϕ : {0, 1}∗ → {0, 1}∗ such that σ ⪯ τ implies ϕ(σ) ⪯ ϕ(τ); Φ(X) is defined when \bigcup_n ϕ(X↾n) is infinite, and in that case they are equal. Similar representations hold for functions on Nᴺ.

An infinite set C ⊆ ω is called immune if it does not include any infinite c.e. subset, or equivalently if it has no infinite computable subset. A c.e. set which is the complement of an immune set is simple.

Definition 1.1. Let P be a closed subset of 2ᴺ.

1. P is immune if TP is immune.

2. P is tree-immune if TP has no infinite computable subtree.

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It is easy to see that an immune closed set must be tree-immune, and both must be special; i.e., have no computable paths. In section 2 we separate all three notions. We also show that the class of separating sets \( S(A,B) \) for any pair of computably inseparable sets \( A \) and \( B \) is immune and that any perfect thin \( \Pi^0_1 \) class is tree-immune. We define the notion of prompt immunity and construct an example of a \( \Pi^0_1 \) class of positive measure which is promptly immune.

In section 3, we consider connections between immunity and Binns’ notion of smallness [5]. We show that every special hyperimmune \( \Pi^0_1 \) class is tree-immune and that every small special \( \Pi^0_1 \) class is immune. In section 4, we consider connections with the Medvedev degrees of difficulty [20, 23]. We show that for closed sets \( P \) and \( Q \), the meet \( P \oplus Q \) is (tree-)immune if and only if both \( P \) and \( Q \) are (tree-)immune, whereas the join \( P \otimes Q \) is (tree-)immune if and only if at least one of \( P \) and \( Q \) are (tree-)immune. We show that for any \( \Pi^0_1 \) class \( P \) with no computable element, there is a non-immune \( \Pi^0_1 \) class \( Q \) with no computable element which is Medvedev reducible to \( P \). In section 5, we show that no immune-free degree cups to any generalized separating class (in the sense of Cenzer and Hinman [10], and hence every immune-free Medvedev degree is non-cuppable.

In section 6, we show that any random closed set (in the sense of [2]) is immune. We also show that any random closed set is not small.

2 Immunity for \( \Pi^0_1 \) classes

We begin with two useful characterizations of immunity.

**Lemma 2.1.** A closed set \( P \) is immune if and only if \( T_P \) has no infinite c.e. subtree.

**Proof.** Certainly if \( T_P \) has an infinite c.e. subtree, then it has an infinite computable subset. For the converse, let \( S \subseteq T_P \) be an infinite computable subset and define the tree \( T \) by

\[
\sigma \in T \iff (\exists \tau \in S)\sigma \preceq \tau.
\]

Then \( T \) is an infinite c.e. subtree of \( T_P \). \( \square \)

Theorem 2.3 shows we cannot ensure that every infinite c.e. tree has an infinite computable subtree.

**Lemma 2.2.** \( P \) is not immune if and only if there is a computable sequence \( \{\sigma_n : n \in \omega\} \) such that \( \sigma_n \in T_P \cap \{0,1\}^n \) for each \( n \).

**Proof.** The reverse implication is immediate. Now suppose that \( C \) is an infinite computable subset of \( T_P \) and enumerate \( C \) as \( \{\tau_0, \tau_1, \ldots\} \). Observe that \( C \) must have arbitrarily long elements and define \( \sigma_n \) to be \( \tau_i \upharpoonright n \), where \( i \) is the least such that \( |\tau_i| \geq n \). \( \square \)
It is clear any immune class is tree-immune, and any tree-immune class is special. The following results show that neither implication reverses.

**Theorem 2.3.** There exists a tree-immune $\Pi^0_1$ class $P$ that is not immune.

**Proof.** Let $S_e$ be the $e$th computable tree, with characteristic function $\varphi_e : \{0,1\}^* \rightarrow \{0,1\}$. We will say that $S_e$ has height $\geq m$ at stage $s$ if $\varphi_{e,s}(\sigma) \text{ is defined for all } \sigma \in \{0,1\}^m$ and $S_e$ has at least one node of length $m$. We will build a sequence of nested computable trees $T_s$ such that $T_P = \bigcap_s T_s$ and a prefix-free infinite c.e. set $A$ such that $A_s = \{\sigma_0, \ldots, \sigma_s\} \subseteq Ext(T_s)$ and $|\sigma_s| > s$. We have the following requirements:

$$N_e : |S_e| = \infty \Rightarrow S_e \notin T_P.$$  

Each $N_e$ has an associated $m_0(e)$, the minimum height of $S_e$ required before we act for $N_e$. For all $e$, $m_0(e) = 2e + 1$.

To meet a single requirement $N_0$ we wait until the stage $s$ when $S_0$ attains height $\geq 1 (= m_0(0))$. Then we choose the leftmost $\tau$ in $S_0 \cap \{0,1\}$, let $m_s(0) = 1 + \max\{|\sigma_i| : i < s\}$, and choose all $\sigma_i$, $t \geq s$, to be incompatible with $\tau$. Then at stage $t > s$ when $S_0$ reaches height $\geq m_s(0)$, we choose $\tau' \in S_e \cap \{0,1\}^{m_s(0)}$ extending $\tau$ and let $T_{t+1}$ be the result of removing from $T_t$ all extensions of $\tau'$. If $S_e$ has no extensions of $\tau$ of length $m_s(0)$, then if $\tau = 1$, or $\tau = 0$ but $1 \notin S_e$, we abandon $N_0$, as $S_e$ is finite. Otherwise we reset $\tau$ to 1 and $m_t(0) = 1 + \max\{|\sigma_i| : i < t\}$ and wait again, avoiding the cone above the new $\tau$ (and no longer avoiding the old) in future $\sigma_i$ choices.

The same module holds for all other requirements; we maintain a set $R$ of bases of cones that must be avoided by $A$. Each $m_s(e)$ changes its value at most $2^{2e+1}$ times, and the values it takes on are sufficiently large that standard measure arguments show we always have room to choose new $\sigma_i$ nodes and maintain their extendibility.

Stage 0: $\forall e \ m_0(e) = 2e + 1$; $A_0 = R_0 = \emptyset$; $T_0 = \{0,1\}^*$. 

Stage $s > 0$: Step 1. For each $e \leq s$ such that $S_e$ has height $\geq 2e + 1$ newly at stage $s$, set $m_s(e) = 2e + 1 + \max\{|\sigma_i| : i < s\}$ and set $\tau_e$ to the leftmost string in $S_e \cap \{0,1\}^{e+1}$. Enumerate all such $\tau_e$ into $R_s$.

Step 2. For each $e \leq s$ such that $m_{s-1}(e) > 2e + 1$, $S_e$ has height $m_{s-1}(e)$ newly at $s$, and $S_e \cap \{0,1\}^{m_{s-1}(e)} \subseteq T_{s-1}$, if there exists a string $\tau > \tau_e$ in $S_e \cap \{0,1\}^{m_{s-1}(e)}$ remove the leftmost such from $T_s$. If there does not exist such a $\tau$, remove $\tau_e$ from $R_s$. If $\tau_e$ is the rightmost string in $S_e \cap \{0,1\}^{2e+1}$, do nothing. Otherwise choose the leftmost of the strings to the right of $\tau_e$, label it the new $\tau_e$, put this new $\tau_e$ into $R_s$, and set $m_s(e) = 2e + 1 + \max\{|\sigma_i| : i < s\}$.

Step 3. For any $e$ not treated above, let $m_s(e) = m_{s-1}(e)$; let $T_s$ be $T_{s-1}$ minus the strings removed in the previous step (if any) and all their extensions.
Step 4. Finally, let $Q$ be the part of $T_s$ uncovered by $A$ and $R$. That is,

$$Q = T_s - \{ \tau^\rho : \tau \in A_{s-1} \cup R_s, \rho \in \{0, 1\}^* \}.$$

Note that since we only remove strings from $T$ that are within the intervals of permanent members of $R$, we would get the same $Q$ if we replaced $T_s$ with $\{0, 1\}^*$. Choose the leftmost $\sigma \in Q$ of length at least $s + 2$ and let it be $\sigma_s \in A_s$.

To verify the construction works, first note every $\sigma_i$ has an extension by a straightforward measure argument: we remove at most one node $\tau$ on behalf of each $S_\epsilon$, and for any $i$ such that $\tau \geq \sigma_i$, we ensure $\mu(I(\tau)) \leq 2^{-2^{e-1-|\sigma_i|}}$. The sum of the measure removed from any $I(\sigma_i)$ is hence bounded by $\frac{2}{3} \mu([\sigma_i])$.

Another measure argument shows there is always enough room in $Q$ to choose a new string in $A$ without covering all of $T_s$. Since each $S_\epsilon$ has at most one node in $R$ at a time, the measure of $Q$ at stage $s$ is at least

$$x = 1 - \sum_{e=0}^{s} 2^{-2^{e-1}} - \sum_{i=1}^{s-1} 2^{-i-2},$$

which we need to be greater than (at most) $2^{-s-2}$. It is easily checked that $x - 2^{-s-2}$ is

$$\frac{1}{12} + \frac{1}{3} \cdot 2^{2s+1} + \frac{1}{2s+2},$$

which is clearly positive.

Since it is clear that the requirements are met, $P$ is a $\Pi^0_1$ class, and $A \subset T_P$ is computable, the proof is complete.

**Theorem 2.4.** There is a special $\Pi^0_1$ class that is not tree-immune.

*Proof.* This is a corollary of Theorem 4.8; any $Q^*$ where $Q$ is special is also special but not tree-immune.

The next results show many $\Pi^0_1$ classes of interest are immune. Recall $S(A, B)$ denotes the class of separating sets for $A$ and $B$ (all $C$ such that $A \subseteq C$ and $B \cap C = \emptyset$); it is a closed set, and when $A$ and $B$ are c.e. it is a $\Pi^0_1$ class.

**Proposition 2.5.** If $A$ and $B$ are computably inseparable, then $S(A, B)$ is immune.

*Proof.* Suppose that $W \subset T_{S(A, B)}$ is an infinite c.e. set, enumerated without repetition as $\sigma_0, \sigma_1, \ldots$. Note that for any $\sigma \in W$ and any $n < |\sigma|$, $n \in A \Rightarrow \sigma(n) = 1$ and $n \in B \Rightarrow \sigma(n) = 0$. Since $W$ must have elements of arbitrary length, we may computably define $i(n)$ to be the least $i$ such that $|\sigma_i| > n$, and let $X(n) = \sigma_{i(n)}(n)$ to compute a separating set for $A$ and $B$. 

\[ \square \]
The notion of a thin $\Pi^0_1$ class corresponds to that of a hyperhyperimmune set and has been studied extensively by many researchers in articles including [8, 11, 15]. A $\Pi^0_1$ class $P$ is thin if for any $\Pi^0_1$ class $Q \subseteq P$, there is a clopen set $U$ such that $Q = P \cap U$. This is equivalent to saying that the family of $\Pi^0_1$ subsets of $P$ is complemented, that is, for any $\Pi^0_1$ class $Q \subseteq P$, $P \setminus Q$ is also a $\Pi^0_1$ class. Since any hyperhyperimmune set is also immune, the following result is natural.

**Proposition 2.6.** If $P$ is a perfect thin $\Pi^0_1$ class, then $P$ is tree-immune.

**Proof.** Let $P$ be perfect thin (and therefore having no computable member) and suppose that some infinite computable tree $W \subseteq T_P$. Let $L$ be the set of leaves of $W$, that is

$$L = \{ \sigma \in W : \sigma \not\in W & \sigma \not\in W \}.$$ 

Then the elements of $L$ are pairwise incomparable and, since $P$ has no computable elements, $L$ is infinite. To see this, note that if $L$ were finite, then $Ext(W)$ would be computable and thus $W$ would have a computable element (in particular the leftmost path), which would also belong to $P$. That is, suppose that $L$ were finite and let $m$ be the maximum length of a node in $L$, then, for any $\sigma$, 

$$\sigma \in Ext(W) \iff (\exists \tau \in W \cap \{0,1\}^{m+1}) \sigma \prec \tau.$$ 

Note that for each $\sigma \in L$, $\sigma \in T_P$. Now we can partition $P$ into the following subsets:

$$P_0 = \{ X \in P : (\forall n) X[n \notin L] \}$$

and

$$P_1 = \{ X \in P : (\exists n) X[n \in L] \}.$$ 

$P_0$ is a $\Pi^0_1$ class and therefore, since $P$ is thin, $P_1$ is also a $\Pi^0_1$ class.

Let $L = \{ \sigma_0, \sigma_1, \ldots \}$ and observe that the closed set $P_1$ is covered by the family $\{ I(\sigma_i) : i \in \omega \}$. It follows by compactness that $P_1 \subseteq I(\sigma_0) \cup \cdots \cup I(\sigma_k)$ for some finite $k$. But this contradicts the fact that every $\sigma_i \in T_P$ and that the $\sigma_i$s are pairwise incomparable. 

A c.e. set $A$ is called promptly simple if for some enumeration $\{ A_n \}_{n \in \mathbb{N}}$ of $A$ there is a computable function $\pi$ such that for any infinite c.e. set $W_e \subseteq \mathbb{N}$ there are $n, s$ with $n \in W_{e^{s+1}} - W_{e^s}$ and $n \in A_{\pi(s)}$.

For $P$ a $\Pi^0_1$ class, let $T$ be a computable tree giving $P$. For each $s$, let $T_s$ be the collection of nodes of $T$ which have length-$s$ extensions in $T$. Let $\{ \sigma_n \}_{n \in \mathbb{N}} = \{ \lambda, 0, 1, 00, 01, 10 \ldots \}$ denote the length-lexicographical ordering of the
elements of \( \{0,1\}^* \). We say that \( P \) is promptly immune if there is a computable function \( \pi \) such that for any infinite c.e. set \( W \), there exist \( n, s \) such that

\[ n \in W_{s+1} - W_s \& \sigma_n \notin T_{\pi(s)}. \]

There exist \( \Pi_1^0 \) classes with positive measure which have no computable elements. The next result is an improvement on this.

**Theorem 2.7.** There exists a \( \Pi_1^0 \) class \( P \) of positive measure which is promptly immune.

**Proof.** We define the \( \Pi_1^0 \) class \( P = [T] \) in stages \( T_s \) and let \( T = \bigcap_s T_s \). \( P \) will be promptly immune via the function \( \pi(s) = s + 1 \). For each \( e \), we will wait for some \( n \) such that \( |\sigma_n| > 2e \) to come into \( W_e \) at stage \( s+1 \) and then remove \( \sigma_n \) from \( T_{s+1} \) by removing \( \sigma_n \) and all extensions (if any) from \( T \). Initially \( T_0 = \{0,1\}^* \). After stage \( s \), we will have satisfied some of the requirements. At stage \( s+1 \), we look for the least \( e \leq s \) which has not yet been satisfied and such that some suitable \( n \in W_{e,s+1} - W_{e,s} \). We meet this requirement by setting

\[ T_{s+1} = T_s - \{ \tau : \sigma_n \preceq \tau \}. \]

Note that this action removes from \( [T] \) a set of measure \( \leq 2^{-2e-1} \), so that the total measure removed is

\[ \leq \sum_e 2^{-2e-1} = \frac{2}{3}. \]

It follows that \( T_s \neq \emptyset \) for any \( s \) and therefore \( P = [T] \) is not empty, and in fact has measure at least \( \frac{1}{3} \).

\[ \Box \]

3 Smallness and Hyperimmunity

In this section, we compare immunity with other “smallness” notions for \( \Pi_1^0 \) classes. Some definitions are needed.

There is a one-to-one correspondence between the set of natural numbers and the set of finite subsets of natural numbers, given as follows. For any \( n > 0 \), let \( n \) be uniquely expressed in binary form as \( n = \sum_{j=1}^{k} 2^{e_j} \) for some finite sequence \( e_1 < e_2 < \cdots < e_k \); the finite set \( \{e_1, \ldots, e_k\} \) is denoted by \( D_n \) and \( n \) is its canonical index. We set \( D_0 = \emptyset \). For any computable function \( f \), the sequence \( D_{f(n)} \) is called a strong array; it is called disjoint if the sets \( D_{f(n)} \) are pairwise disjoint.

A set \( C \subseteq \mathbb{N} \) is called hyperimmune if there is no disjoint strong array \( \langle D_{f(n)} \rangle \) such that, for all \( n \), \( D_{f(n)} \cap C \neq \emptyset \). A well-known theorem by Kuznecov, Medvedev, and Uspenski ([25] V.2.3) states that \( C = \{c_0 < c_1 < \ldots \} \) is hyperimmune if and only if there is no infinite computable function \( g \) such that \( g(n) > c_n \) for all \( n \).
A finite string $\sigma \in \{0, 1\}^n$ has Gödel number $\sum_{i=0}^{n} \sigma(i)2^i$. If $F$ is a finite set of (Gödel numbers of) strings, then $F^* = \bigcup \{I(\sigma) : \sigma \in F\}$. Binns [6] called a sequence $(D_f(n))$ of finite sets of (Gödel numbers of) strings a disjoint strong array if the sets $D_f^*(n)$ are pairwise disjoint.

**Definition 3.1.**

1. (Binns [5]). A closed set $P$ is **small** if there is no computable function $g$ such that, for all $n$, $\text{card}(\{0, 1\}^g(n) \cap T_P) > n$.

2. (Binns [6]). A closed set $P$ is **hyperimmune** if there is no disjoint strong array $(D_f(n))$ such that $P \cap D_f^*(n) \neq \emptyset$ for all $n$.

Binns [6] showed that the class $\text{DNC}_2$ of diagonally non-computable functions is not small, and in fact not hyperimmune. By Proposition 2.5, this gives an example of an immune class of measure 0 which is not small. It is also easy to see that a class of positive measure cannot be small, so the immune class of Theorem 2.7 is also not small.

For any tree $T \subseteq \{0, 1\}^*$, we say that $\sigma$ is a branching node of $T$ if both $\sigma \ddash 0$ and $\sigma \ddash 1$ are in $T$; let $\text{Br}(T)$ denote the set of branching nodes of $T$.

**Theorem 3.2 (Binns [5]).** A $\Pi^0_1$ class $P$ is small if and only if $\text{Br}(T_P)$ is hyperimmune.

**Theorem 3.3 (Binns [6]).** Every small $\Pi^0_1$ class is hyperimmune.

The converse to Theorem 3.3 does not hold. It is not clear whether every special hyperimmune $\Pi^0_1$ class must be immune, because the nodes witnessing immunity need not be incomparable. However, we have the following result.

**Theorem 3.4.** Every special hyperimmune $\Pi^0_1$ class is tree-immune.

*Proof.* Assume $P$ is not tree-immune, and let $T \subseteq T_P$ be a computable tree. Since $P$ is special, $T$ has an infinite, computable set $L = \{\sigma_0, \sigma_1, \ldots\}$ of leaves. Then we may define a disjoint strong array

$$D_f(n) = \{\sigma_n\}.$$ 

Hence $P$ is not hyperimmune. \hfill $\square$

Cenzer, Weber, and Wu [14] asked whether every small special $\Pi^0_1$ class is immune. We can now answer this question.

**Theorem 3.5.** Every small special $\Pi^0_1$ class is immune.

*Proof.* Suppose that $P$ is special and small but not immune, and let $T \subseteq T_P$ be an infinite c.e. subtree.
Claim 3.6. \( Br(T) \) is infinite.

*Proof.* Suppose by way of contradiction that \( Br(T) \) is finite and let \( s \) be the maximum length of any \( \sigma \in Br(T) \). It follows that any node in \( T \) of length \( \geq s \) must extend one of the finite set of nodes of length \( s \). Since \( T \) is infinite, there must be a single node \( \tau \in T \) which has infinitely many extensions in \( T \). Since \( T \) is c.e., we may compute the path \( X \) as follows. Given \( i \), enumerate the elements of \( T \) until we find a string \( \sigma \) with \( |\sigma| \geq i \) which is comparable with \( \tau \) and then \( X(i) = \sigma(i) \). This violates the assumption that \( P \) is special. \( \square \)

Now \( Br(T) \) is itself a c.e. set, since we can enumerate \( \sigma \in Br(T) \) once \( \sigma, \sigma \vdash 0 \), and \( \sigma \vdash 1 \) have all been enumerated into \( T \). Hence \( Br(T) \) has an infinite, increasing computable subset and is certainly not hyperimmune. It follows that the larger set \( Br(T_P) \) is also not hyperimmune, so by Theorem 3.2 \( P \) is not small. \( \square \)

4 Degrees of Difficulty

\( \Pi_0^1 \) classes are often viewed as collections of solutions to some mathematical problem. Muchnik and Medvedev reducibility, defined for closed subsets of \( 2^N \) and indeed \( N^N \) in general, order classes based on this viewpoint. The class \( A \) is Muchnik (a.k.a. weakly) reducible to the class \( B \) \( (A \leq_w B) \) if for every \( X \in B \) there is \( Y \in A \) such that \( Y \leq_T X \) [21]. The class \( A \) is Medvedev (a.k.a. strongly) reducible to \( B \) \( (A \leq_s B) \) if there is a single Turing reduction procedure which, when given any element of \( B \) as an oracle, computes an element of \( A \); it is exactly the uniformization of Muchnik reduction [20]. These reductions have been studied extensively by Binns (e.g., [4]), Cenzer and Hinman [9, 10] and Simpson (e.g., [24]) and have connections to randomness [22]. We will need the result from [9] that any partial computable \( \Phi : P \to Q \) for two \( \Pi_0^1 \) classes \( P \) and \( Q \) may be extended to a total computable functional. The Medvedev degrees are equivalence classes under \( P \equiv_s Q \), defined as \( (P \leq_s Q) \& (Q \leq_s P) \), and similarly for the Muchnik degrees. Let \( P_s \) denote the partial ordering of the Medvedev degrees of \( \Pi_0^1 \) classes.

**Proposition 4.1.** If \( P \) is not (tree-)immune and \( Q \) is Medvedev reducible to \( P \) then \( Q \) is also not (tree-)immune.

*Proof.* Let \( P \) be a \( \Pi_0^1 \) class which is not tree-immune, and \( V \subseteq T_P \) an infinite computable tree. Let \( \Phi \) witness \( Q \leq_s P \) and set \( S = \Phi(V) \); note that \( S \) is a tree. By the definition of partial computable functional and the fact that \( \Phi \) must be defined on all of \( P \), \( S \subseteq T_Q \) and \( S \) is infinite. It remains to show \( S \) is computable.
To determine whether $\tau \in S$, compute $\varphi(\sigma)$ for all $\sigma \in 2^{<\omega}$ in lexicographical order until $|\varphi(\sigma)| \geq |\tau|$ for all $\sigma \in P$ of some length $n$. Then $\tau \in S$ if and only if $\tau \preceq \varphi(\sigma)$ for some $\sigma \in V \cap \{0,1\}^n$.

If $P$ is not immune, then there is an infinite c.e. tree $V \subseteq T_P$ and the argument above shows that $\Phi(V)$ is an infinite c.e. subtree of $T_Q$, so that $Q$ is also not immune. \hfill \square

Let us say that a Medvedev degree $d \in \mathcal{P}_s$ is (tree-)immune if there is some class $P \in d$ which is (tree-)immune and otherwise $d$ is (tree-)immune-free.

**Corollary 4.2.** 1. If $d \in \mathcal{P}_s$ contains a non-(tree-)immune $\Pi^0_1$ class, then $d$ is (tree-)immune-free. 2. If $d \in \mathcal{P}_s$ contains a (tree-)immune $\Pi^0_1$ class, then every member of $d$ is (tree-)immune.

For $X, Y \in 2^\mathbb{N}$, the join $X \oplus Y = Z$ is given by $Z(2n) = X(n)$ and $Z(2n+1) = Y(n)$. Similarly, for finite sequences $\sigma$ and $\tau$ of equal length, we may define $\sigma \oplus \tau = \rho$, where $\rho(2n) = \sigma(n)$ and $\rho(2n+1) = \tau(n)$.

The quotient structure of the $\Pi^0_1$ classes under either Muchnik or Medvedev equivalence is a lattice, and both have the same join and meet operators. The join of $P$ and $Q$ is given by

$$P \otimes Q = \{X \oplus Y : X \in P, Y \in Q\}.$$  

If $P = [S]$ and $Q = [T]$, then $P \otimes Q = [S \otimes T]$, where

$$S \otimes T = \{\sigma \oplus \tau, (\sigma \oplus \tau)i : \sigma \in S, \tau \in T, |\sigma| = |\tau|, i \in \{0,1\}\};$$

since all finite joins are of even length, we branch at odd levels. The meet of $P$ and $Q$ is given by

$$P \ominus Q = \{0^\sim X : X \in P\} \cup \{1^\sim Y : Y \in Q\}.$$  

If $P = [S]$ and $Q = [T]$, then $P \ominus Q = [S \ominus T]$, where

$$S \ominus T = \{0^\sim \sigma : \sigma \in S\} \cup \{1^\sim \tau : \tau \in T\}.$$  

Binns [6] showed that $P \ominus Q$ and $P \otimes Q$ are small if and only if both $P$ and $Q$ are small. The results for immunity are not quite the same.

**Theorem 4.3.** For any closed sets $P$ and $Q$, $P \oplus Q$ is (tree-)immune if and only if both $P$ and $Q$ are (tree-)immune.
Proof. Suppose first that $P$ is not immune and let $C \subseteq T_P$ be an infinite computable set. Then $\{0^\sigma : \sigma \in C\}$ is a computable subset of $T_{P \otimes Q}$. Suppose $P$ is not tree-immune, let $V \subseteq T_P$ be an infinite tree. Then $\{\lambda\} \cup \{0^\sigma : \sigma \in V\}$ is an infinite computable subtree of $T_{P \otimes Q}$. The arguments when $Q$ is not (tree-)immune are, of course, symmetric.

Next suppose that $P \oplus Q$ is not immune and let $C \subseteq T_{P \oplus Q}$ be an infinite computable set. Let $C_i = \{\sigma : i^\sigma \in C\}$ for $i = 0, 1$. Then $C_0 \subseteq T_P$, $C_1 \subseteq T_Q$ and both sets are computable. Clearly either $C_0$ is infinite or $C_1$ is infinite, which implies that either $P$ is not immune or $Q$ is not immune. A similar argument applies if $P \otimes Q$ is not tree-immune, where $V$, $V_P$ and $V_Q$ are all infinite c.e. trees.

Theorem 4.4. For any closed sets $P$ and $Q$, $P \otimes Q$ is (tree-)immune if and only if at least one of $P$ and $Q$ is (tree-)immune.

Proof. Suppose first that $P \otimes Q$ is not tree-immune and let $V \subseteq T_{P \otimes Q}$ be an infinite computable tree. Let $V_P = \{\sigma : (\exists \tau \in \{0, 1\}^{[\sigma]})(\sigma \oplus \tau \in V)\}$ and similarly $V_Q = \{\tau : (\exists \sigma \in \{0, 1\}^{[\tau]})(\sigma \oplus \tau \in V)\}$. Then $V_P$ is an infinite computable subtree of $T_P$ and $V_Q$ is an infinite computable subtree of $T_Q$, so that neither $P$ nor $Q$ is tree-immune. A similar argument applies if $P \otimes Q$ is not immune, where $V$, $V_P$ and $V_Q$ are now infinite c.e. trees.

Next suppose that both $P$ and $Q$ are not tree-immune and let $V_P \subseteq T_P$ and $V_Q \subseteq Q$ be infinite computable trees. Then $V_P \otimes V_Q$ is an infinite computable subtree of $T_P \otimes T_Q = T_{P \otimes Q}$. A similar argument applies if $P$ are $Q$ are both not immune, where $V_P$, $V_Q$ and $V_P \otimes V_Q$ are all infinite c.e. trees.

Corollary 4.5. The immune-free degrees and the tree-immune-free degrees each form a prime ideal in the lattice $P_\infty$.

Corollary 4.6. The tree-immune-free Medvedev degrees form a proper subideal of the immune-free Medvedev degrees.

Proof. Let $d$ be the Medvedev degree of the tree-immune, non-immune $\Pi^0_1$ class $P$ constructed in Theorem 2.3. Then by Corollary 4.2, $d$ is tree-immune but immune-free.

We now turn to questions of density. Let $0_\infty$ denote the least Medvedev degree, which consists of all $\Pi^0_1$ classes that have a computable member. Binns has shown there is a nonsmall class of every nonzero Medvedev degree. We have the following bounding result for nonimmune classes.
**Theorem 4.7.** For any nonzero $\Pi^0_1$ class $P$, there is a $\Pi^0_1$ class $Q$ with $0_s <_s Q \leq_s P$ which is not tree-immune, and hence not immune.

*Proof.* Let $R$ be the $\Pi^0_1$ class of Theorem 2.4 which is nonzero and not tree-immune. It follows from Theorem 4.3 that $P \oplus R$ is not tree-immune, but it is also special and certainly $P \oplus R \leq_s P$. □

**Theorem 4.8.** For every $\Pi^0_1$ class $Q$, there exists a $\Pi^0_1$ class $Q^*$ such that $Q^*$ has tree-immune-free Medvedev degree, and $Q^*$ is Muchnik equivalent to $Q$. Furthermore, if $Q$ is immune, then $Q^* <_s Q$.

*Proof.* The case that $Q$ is not special is obvious. Let $Q$ be a special $\Pi^0_1$ class, and let $T$ be a computable tree such that $[T] = Q$. We note that the set $L$ of all leaves of $T$ is computable. We set

$$T^* = T \cup \{\sigma^\tau : \sigma \in L \& \tau \in T\},$$

and let $Q^* = [T^*]$, so that

$$Q^* = Q \cup \{\sigma^X : \sigma \in L \& X \in Q\}.$$  

Then $Q^*$ is a $\Pi^0_1$ class and $Q \subseteq Q^*$, so $Q^* \leq_s Q$. $T$ is a computable subtree of $T_Q^*$, so that $Q^*$ is not tree-immune, and hence by Corollary 4.2, $Q^*$ has tree-immune-free degree. At the same time, every member of $Q^*$ is Turing equivalent to a member of $Q$, so that $Q^*$ is Muchnik equivalent to $Q$.

If $Q$ is immune, it follows from Proposition 4.1 that we may not have $Q \leq_s Q^*$, since $Q^*$ is not immune. □

**Lemma 4.9 (Essentially by Simpson [23]).** There exists a Medvedev complete set $Q$ and a computable function $q$ such that, for any $e$, the $e^{th}$ $\Pi^0_1$ class $P_e$ is Medvedev reducible to $Q$ via a computable functional $\Phi_{q(e)}$.

**Remark 4.10.** Every Medvedev complete set has this property.

**Lemma 4.11.** Let $P \leq_s Q$ be special $\Pi^0_1$ classes, $S$ and $T$ computable trees with $[S] = P$ and $[T] = Q$, and $L_S$ and $L_T$ the computable sets of all leaves of $S$ and $T$, respectively. Then there is a computable functional $\Phi^*$ such that $\Phi^*(Q) \subseteq P$ and $\Phi^*(L_T) \subseteq L_S$.

*Proof.* Since $P$ is special, any $\sigma \in S$ has an extension in $L_S$. Assume $P \leq_s Q$ via the computable functional $\Phi$ and let $\varphi$ be a representing function for $\Phi$. We construct the desired functional $\Phi^*$ with representing function $\varphi^*$.

First suppose $\tau \in 2^{<\omega}$ has no initial segment which is a leaf of $T$. If $\varphi(\tau) \in S$, then we let $\varphi^*(\tau) = \varphi(\tau)$. If $\varphi(\tau) \notin S$, then we let $\sigma$ be the longest initial segment of $\varphi(\tau)$ which belongs to $S$, so that $\sigma \in L_S$, and let $\varphi^*(\tau) = \sigma$. Note
that if \( X \in Q \), it follows that \( \varphi^*(X \upharpoonright n) = \varphi(X \upharpoonright n) \), so that \( \Phi^*(Q) \subseteq P \) as desired.

Next suppose that \( \tau \succeq \sigma \) for some leaf \( \sigma \) of \( T \). If \( \varphi(\sigma) \notin S \), then as above let \( \varphi^*(\sigma) \) be the longest initial segment of \( \varphi(\sigma) \) which belongs to \( S \). If \( \varphi(\sigma) \in S \), let \( \varphi^*(\sigma) \) be the shortest and leftmost leaf of \( S \) which extends \( \varphi(\sigma) \). Then let \( \varphi^*(\tau) = \varphi^*(\sigma)^{0_{|\tau|-|\sigma|}} \). It follows that \( \varphi^* \) maps \( L_T \) into \( L_S \).

It is easy to check that \( \varphi^* \) is monotonic and defines a computable functional \( \Phi^* \).

\[ \square \]

**Theorem 4.12.** There is a greatest tree-immune-free Medvedev degree.

**Proof.** Let \( Q \) be a Medvedev complete set, \( T \) a computable tree such that \( Q = [T] \) and \( Q^* \) as defined in Theorem 4.8. Fix any non-tree-immune \( \Pi^0_1 \) class \( P \) and let \( V \subseteq T_P \) be an infinite computable tree. We may assume that \( P \) has no computable path. By the Medvedev completeness of \( Q \) and Lemma 4.11, \( [V] \leq_s Q \) via some computable functional \( \Phi \) with representing function \( \varphi \) such that \( \varphi \) maps \( L_T \) into \( L_V \).

Let \( f \) be a computable function such that \( P_{f(\sigma)} = P \cap I(\sigma) \) for all \( \sigma \in L_V \), and observe that since \( V \subseteq T_P \), \( P_{f(\sigma)} \) is a nonempty subset of \( P \).

We now construct a computable functional \( \Psi : Q^* \rightarrow P \). Let \( X \in Q^* \). We define the partial output \( \psi(X \upharpoonright n) \) as follows. As long as \( \varphi(X \upharpoonright n) \in V \), simply let \( \psi(X \upharpoonright n) = \varphi(X \upharpoonright n) \). If \( \varphi(X \upharpoonright n) \in V \), but \( \varphi(X \upharpoonright n + 1) \notin V \), then there exists \( \sigma \in L_V \) with \( \varphi(X \upharpoonright n) \preceq \varphi(X \upharpoonright n + 1) \). Furthermore, since \( \Phi : Q \rightarrow [V] \) and \( \varphi(X \upharpoonright n + 1) \notin V \), it follows that \( X \notin Q \). In this case, it follows by the assumption from Lemma 4.11 that \( X \upharpoonright n + 1 \in L_V \). To see this, let \( k \) be the least such that \( X \upharpoonright k \in L_V \). Then \( \varphi(X \upharpoonright k) = \sigma \) by the assumption from Lemma 4.11 and the monotonicity of \( \varphi \). Also \( k \leq n \) since \( \varphi(X \upharpoonright n + 1) \notin V \) and hence \( \varphi(X \upharpoonright n) = \sigma \) as well.

Now define \( \Psi(X) = \Phi_{\Phi_f(\sigma)}(X) \), where \( q \) is the function from Lemma 4.9. Since we know \( \sigma \prec \Phi_{\Phi_f(\sigma)}(X) \), we can let \( \psi(X \upharpoonright n + r) = \sigma \cup \varphi_{\Phi_f(\sigma)}(X \upharpoonright n + r) \), that is, \( \psi(X \upharpoonright n + r) = \sigma \) if \( \varphi_{\Phi_f(\sigma)}(X \upharpoonright n + r) \preceq \sigma \) and otherwise \( \psi(X \upharpoonright n + r) = \varphi_{\Phi_f(\sigma)}(X \upharpoonright n + r) \).

\[ \square \]

**Corollary 4.13.** The c-immune-free Medvedev degrees forms a principal prime ideal in \( P_s \).

## 5 Non-Cupping

Cenzer-Weber-Wu [14] suggested the problem of determining the cuppable \( \Pi^0_1 \) classes in \( P_s \). Here we say that an incomplete \( \Pi^0_1 \) class \( P \) is *cuppable* if there exists an incomplete \( \Pi^0_1 \) class \( Q \) such that \( P \otimes Q \) is Medvedev complete. In general, \( P \) *cups* to \( R \) if there exists \( Q \leq_s R \) such that \( P \otimes Q \equiv_s R \).
The first result in this direction is the following.

**Theorem 5.1 (Simpson [23]).** Any Π^0_1 class that cups to a separating class must have measure 0.

Hence, the positive measure Medvedev degrees POS form a subideal of Medvedev non-cupping degrees NCup, and, by Theorem 2.7, a non-cuppable promptly immune Π^0_1 class exists. However, we will observe a further relationship between immunity and non-cuppability.

Recall for disjoint sets A, B, S(A, B) is the class of all separating sets C ⊇ A, C ∩ B = ∅. In particular, DNC_2 = S(A_0, A_1) where A_i = {e : ϕ_e(e) = i}. A *generalized separating class* is the product \( \prod_n F_n \) where \( \{F_n\}_{n \in \omega} \) is a computable sequence of finite subsets of \( \mathbb{N} \). For S(A, B) the set \( F_n = \{0\} \) if \( n \in B \), \( \{1\} \) if \( n \notin A \cup B \). Generalized separating classes were studied by Cenzer and Hinman [10]. It is important to note that any generalized separating class \( P \) is computably bounded and hence is computably homeomorphic to a Π^0_1 class \( Q \subseteq \{0, 1\}^\omega \) (see Lemma 1.3 of [7]). Hence the Medvedev degrees of the generalized separating classes are included in the Medvedev degrees of subsets of Cantor space.

**Theorem 5.2.** No immune-free degree cups to any generalized separating class.

*Proof.* Let \( P \) be an non-immune Π^0_1 class and \( V \subseteq T_P \) an infinite computable set, with fixed enumeration \( \{\sigma_i\}_{i \in \omega} \). Let \( S = \prod_n F_n \) be a generalized separating class for a sequence \( \{F_n\}_{n \in \omega} \) of finite sets. Suppose that for some \( Q \), \( S \leq P \otimes Q \) via a computable functional \( \Phi \). We will write an input \( X \oplus Y \) to \( \Phi \) as the ordered pair \( X, Y \).

We construct a computable functional \( \Psi \) witnessing \( S \leq_s Q \). Given \( Y \in Q \), define \( Z = \Psi(Y) \) as follows. For each \( n \), let \( Z(n) = \Phi(\sigma_i, Y)(n) \), where \( i \) is the least such that \( \Phi(\sigma_i, Y)(n) \) is defined. We know that such \( i \) exists since, by compactness, there is some \( m \) such that \(|\varphi(\sigma, \tau)| > n \) for all \( \sigma \in T_P, \tau \in T_Q \) with length \( \geq m \).

It remains to confirm that \( Z = \Psi(Y) \in S \); that is, \( Z(n) \in F_n \) for all \( n \). Given \( n \) and \( \sigma_i \in V \) such that \( Z(n) = \Phi(\sigma_i, Y)(n) \), we can find \( X \in P \) such that \( \sigma_i \prec X \) (since \( \sigma_i \in T_P \)). It follows that \( \Phi(X, Y) \in S \) and hence \( \Phi(\sigma_i, Y)(n) = \Phi(X, Y)(n) \in F_n \).

**Corollary 5.3.** Every immune-free Medvedev degree is Medvedev non-cuppable.

*Proof.* The class of 2-valued diagonally noncomputable functions, DNC_2, is a Medvedev complete generalized separating class, and hence no immune-free degree can cup to it.
We get new subideals $\overline{\text{IM}}$ and $\overline{\text{TIM}}$ of Medvedev non-cuppable degrees $\text{NCup}$, which consist of immune-free and tree-immune-free degrees, respectively. However, immunity does not necessarily give a cupping property. Actually, as seen before, a positive measure promptly immune degree in Cenzer-Weber-Wu [14] is an example of a non-cuppable immune degree.

**Corollary 5.4.** A Muchnik complete Medvedev non-cuppable degree exists.

**Proof.** By Theorem 4.12, $\max \overline{\text{TIM}}$ exists and it is clearly Muchnik complete since it is degree-isomorphic to any Medvedev complete class. Moreover, it is non-cuppable by Corollary 5.3.

**Theorem 5.5.** For $c = \max \overline{\text{TIM}}$, a measure 0 immune non-cuppable degree $c$ exists.

**Proof.** Let $d$ be a positive measure, promptly immune Medvedev degree. Then $d \not\leq c$ holds since tree-immune-free degrees are downward closed. We claim $a = c \cup d$ is the desired degree. This follows from the results that immune degrees and non-cuppable degrees form ideals, positive measure-free degrees form a filter, and $d$ has positive measure-free degree by its Muchnik completeness (see Simpson [23]).

**Corollary 5.6.** The immune-free Medvedev degrees $\overline{\text{IM}}$ and the tree-immune-free Medvedev degrees $\overline{\text{TIM}}$ form proper subideals of the non-cuppable Medvedev degrees $\text{NCup}$.

### 6 Immunity and randomness

Finally we consider the immunity of random closed sets. A closed set $P$ may be coded as an element of $3^\mathbb{N}$; $P$ is called random if that sequence is Martin-Löf random (for background on randomness see [16]). The code of $P$ is defined from $T_P$; the nodes of $T_P$ are considered in order by length and then lexicographically, and each one is represented in the code by 0, 1, or 2 according to whether the node has only the left child, only the right child, or both children, respectively. Randomness for closed sets is defined and explored in [2, 3], where it is shown among other results that no $\Pi^0_1$ class is random, and that no random closed set contains an $f$-c.e. path for any computable $f$ bounded by a polynomial. The following theorem does not follow immediately but is not surprising.

**Theorem 6.1.** If $P$ is a random closed set, then $P$ is immune.

**Proof.** Fix a computable sequence $C = (\sigma_1, \sigma_2, \ldots)$ such that $|\sigma_n| = n$ for each $n$. For $n > 0$, let $S_n = \{Q : (\forall i \leq n) \sigma_i \in T_Q\}$. Then $S_n$ is a clopen set in
the space of closed sets and the sequence \( \{S_n : n \in \omega \} \) is uniformly c.e. It is clear that \( C \subseteq T_P \) if and only if \( P \in S_n \) for all \( n \). Now consider the Lebesgue measure \( \mu(S_n) \). Certainly \( \mu(S_1) = 2/3 \). Given \( \mu_n = \mu(S_n) \) and \( \sigma_{n+1} \), let \( i \leq n \) be the largest such that \( \sigma_i \prec \sigma_{n+1} \). Then \( \mu_{n+1} = \left( \frac{2}{3} \right)^{n+1-i} \mu_n \leq \frac{2}{3} \mu_n \). Hence \( \mu(S_n) \leq \left( \frac{2}{3} \right)^n \) for each \( n \). It follows that \( \{S_{2n} : n \in \omega \} \) is a Martin-Löf test and hence no random closed set can belong to every \( S_n \). Hence if \( P \) is random, \( C \) is not a subset of \( T_P \). Since this holds for every such \( C \), it follows that random closed sets are immune.

Since a random ternary sequence must contain \( \frac{1}{3} \) 2s in the limit, intuitively the tree it codes must branch too much to be small. This is a straightforward consequence of the following, which is drawn from Lemma 4.5 in [2].

**Lemma 6.2.** Let \( Q \) be a random closed set. Then there exist a constant \( C \in \mathbb{N} \) and \( k \in \mathbb{N} \) such that for all \( m > k \),

\[
C \left( \frac{4}{3} \right)^m \left( 1 - m^{-\frac{1}{4}} \right) < \text{card}(T_Q \cap \{0,1\}^m) < C \left( \frac{4}{3} \right)^m \left( 1 + m^{-\frac{1}{4}} \right).
\]

**Corollary 6.3.** If \( Q \) is a random closed set, \( Q \) is not small.

**Proof.** For \( C, k \) as in Lemma 6.2, define the function \( g(n) \) as

\[
g(n) = \max \left\{ k + 1, \min \left\{ m : n < C \left( \frac{4}{3} \right)^m \left( 1 - m^{-\frac{1}{4}} \right) \right\} \right\}.
\]

It is clear that \( g \) is computable, and by Lemma 6.2, for all \( n \) the number of branches at level \( g(n) \) will be at least \( n \).

**References**


REFERENCES


REFERENCES


