Σ_1^0 and Π_1^0 equivalence structures*

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Abstract

We study computability theoretic properties of Σ_1^0 and Π_1^0 equivalence structures and how they differ from computable equivalence structures or equivalence structures that belong to the Ershov difference hierarchy. Our investigation includes the complexity of isomorphisms between Σ_1^0 equivalence structures and between Π_1^0 equivalence structures.

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1 Introduction

Computable model theory deals with the algorithmic properties of effective mathematical structures and the relationships between such structures. Perhaps the most basic kind of relationship between two structures is that of

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isomorphism. It is natural to study the isomorphism problem in the context of computable mathematics by investigating the following question.

Given two effective structures which are isomorphic, what is the least complex isomorphism between them?

In what follows, we restrict our attention to countable structures for computable languages. Hence, if a structure is infinite, we can assume that its universe is the set of natural numbers, ω. We recall some basic definitions. If $\mathcal{A}$ is a structure with universe A for a language $\mathcal{L}$, then $\mathcal{L}^A$ is the language obtained by expanding $\mathcal{L}$ by constants for all elements of A. The *atomic diagram* of $\mathcal{A}$ is the set of all quantifier-free sentences of $\mathcal{L}^A$ true in $\mathcal{A}$. The *elementary diagram* of $\mathcal{A}$ is the set of all first-order sentences of $\mathcal{L}^A$ true in $\mathcal{A}$. A structure $\mathcal{A}$ is *computable* if its atomic diagram is computable, and a structure $\mathcal{A}$ is *decidable* if its elementary diagram is computable. We call two structures *computably isomorphic* if there is a computable function that is an isomorphism between them. A computable structure $\mathcal{A}$ is *relatively computably isomorphic* to a possibly noncomputable structure $\mathcal{B}$ if there is an isomorphism between them that is computable in the atomic diagram of $\mathcal{B}$. A computable structure $\mathcal{A}$ is *computably categorical* if every computable structure that is isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$. A computable structure $\mathcal{A}$ is *relatively computably categorical* if every structure that is isomorphic to $\mathcal{A}$ is relatively computably isomorphic to $\mathcal{A}$. Similar definitions arise for other naturally definable classes of structures and their isomorphisms. For example, for any $n \in \omega$, a structure is Δ^0_n if its atomic diagram is Δ^0_n; two structures are Δ^0_n *isomorphic* if there is a Δ^0_n isomorphism between them; and a computable structure $\mathcal{A}$ is Δ^0_n *categorical* if every computable structure that is isomorphic to $\mathcal{A}$ is Δ^0_n isomorphic to $\mathcal{A}$.

Among the simplest nontrivial structures are equivalence structures, i.e., structures of the form $\mathcal{A} = (\omega, E)$ where E is an equivalence relation. The study of the complexity of isomorphisms between computable equivalence structures was recently carried out by Calvert, Cenzer, Harizanov, and Morozov in [2]. Similarly, the study of structures and functions within the Ershov difference hierarchy has been recently carried out by Khoussainov, Stephan, and Yang in [9], and by Cenzer, LaForte, and Remmel in [3] where they investigated equivalence structures in particular. In this paper, we study Σ^0_1 and Π^0_1 equivalence structures. Here, we say that an equivalence structure $\mathcal{A} = (\omega, E)$ is Σ^0_1 (or *c.e.*) if E is a c.e. set, and, similarly, $\mathcal{A}$ is Π^0_1 (or *co-c.e.*) if E is a Π^0_1 set. It is also the case that Σ^0_1 and Π^0_1 structures have been studied since the beginning of modern computable model theory. For example, in [11], Metakides and Nerode studied c.e. vector spaces, which consist of a structure V over the natural numbers such that the operations of vector addition and scalar multiplication are computable but where there is a c.e. equivalence relation $\equiv$ the equivalences classes of which form a vector space under the vector addition and scalar multiplication. Similarly, in [13], Remmel studied co-c.e. structures where the underlying operations are computable.

Equivalence relations play an important role in mathematical logic and many other areas of mathematics. For example, isomorphism and elementary equivalence, as well as their effective versions such as computable isomorphism or Σ^0_n-equivalence, are equivalence relations. Similarly, a number of interesting applications of equivalence arise from the so-called classification problems where two structures are termed equivalent if they possess certain *invariant* properties.

We shall see that the complexity of isomorphisms between Σ^0_1 equivalence structures and between Π^0_1 equivalence structures is different from the complexity of isomorphisms between computable equivalence structures or between equivalence structures that lie in the Ershov difference hierarchy. Before we can state our results, we need some notation and definitions. For an equivalence structure $\mathcal{A} = (A, E)$ where $A = \omega$, we let $[a]^{\mathcal{A}}$ denote the equivalence class of a, i.e., $[a]^{\mathcal{A}} = \{b \in A : aEb\}$. In computability theory, it is useful to split $\mathcal{A}$ into two parts, $Inf^{\mathcal{A}}$ and $Fin^{\mathcal{A}}$, where $Inf^{\mathcal{A}}$ consists of the elements in infinite equivalence classes, and $Fin^{\mathcal{A}}$ consists of the elements with finite equivalence classes. It is natural to consider different sizes of the equivalence classes of the elements in $Fin^{\mathcal{A}}$ since these sizes code information into the equivalence relation. The *character* of an equivalence structure $\mathcal{A}$ is the set

$$\chi(\mathcal{A}) = \{(k, n) : n, k > 0 \text{ and } \mathcal{A} \text{ has at least } n \text{ equivalence classes of size } k\}.$$

This set provides a kind of skeleton for $Fin^{\mathcal{A}}$. Any set $K \subseteq (\omega - \{0\}) \times (\omega - \{0\})$ such that for all $n > 0$ and k, $(k, n+1) \in K$ implies $(k, n) \in K$, is called a *character*. We say a character K is *bounded* if there is some finite k_0 such that for all $(k, n) \in K$, we have $k < k_0$. Khisamiev [8] introduced the concepts of an s-function and an s_1-function as a means of computably approximating the characters of equivalence relations.

Definition 1.1. *Let $f : \omega^2 \to \omega$. The function f is an s-*function *if the following hold:*

1. for every $i, s \in \omega$, $f(i,s) \leq f(i,s+1)$ *and*
2. for every $i \in \omega$, *the limit* $m_i = lim_s f(i,s)$ *exists.*

We say that f *is an* s_1-function *if, in addition:*
3. for every $i \in \omega$, $m_i < m_{i+1}$.

Calvert, Cenzer, Harizanov and Morozov [2] gave conditions under which a given character K can be the character of a computable equivalence structure. In particular, they observed that if K is a bounded character and $\alpha \leq \omega$, then there is a computable equivalence structure with character K and exactly α infinite equivalence classes. To prove the existence of computable equivalence structures for unbounded characters K, they needed additional information given by s- or s_1-functions. They showed that if K is a Σ_2^0 character, $r < \omega$, and either

(a) there is an s-function f such that

$$(k,n) \in K \Leftrightarrow card(\{i : k = \lim_{s \to \infty} f(i,s)\}) \geq n, \quad \text{or}$$

(b) there is an s_1-function f such that for every $i \in \omega$, $(\lim_s f(i,s), 1) \in K$, then there is a computable equivalence structure with character K and exactly r infinite equivalence classes. In addition to these positive results, in [2] the authors also constructed an infinite Δ_2^0 set D such that for any computable equivalence structure $\mathcal{A}$ with unbounded character and no infinite equivalence classes, $\{k : (k,1) \in K\}$ is not a subset of D.

Σ_1^0 equivalence structures were first considered by Ershov [5] where they are called *positive* equivalence relations. Bernardi and Sorbi [1] referred to Σ_1^0 equivalence structures as *ceers* (computably enumerable equivalence relations) and they developed a notion of reducibility between ceers. Computably isomorphic structures are equivalent under this reducibility but the converse does not hold. This notion was developed further by Gao and Gerdes [6]. C.e. equivalence relations have also been studied by Lachlan [10] and Nies [12].

Definition 1.2. *Let* $\alpha \leq \omega$.

1. *We say the structure* $\mathcal{A}$ *is* weakly α-c.e. isomorphic *to the structure* $\mathcal{B}$ *if there are* α*-c.e. functions* f *and* g *such that* f *is an isomorphism from* $\mathcal{A}$ *to* $\mathcal{B}$ *and* g *is an isomorphism from* $\mathcal{B}$ *to* $\mathcal{A}$.

2. *We say the structure $\mathcal{A}$ is* α-c.e. isomorphic *to the structure $\mathcal{B}$ if there is an α-c.e. function f such that f^{-1} is α-c.e. and f is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

3. *We say the structure $\mathcal{A}$ is* graph-α-c.e. isomorphic *to the structure $\mathcal{B}$ if there is a graph-α-c.e. function f such that f is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, where a function f is* graph-α-c.e. *if the graph of f is an α-c.e. set.*

In [3], Cenzer, LaForte, and Remmel obtained the following results. First they proved the following basic properties of α-c.e. and graph-α-c.e functions.

(a) Any nonempty Σ_2^0 set is the range of a 2-c.e. function.

(b) For every $n \in \omega$, there is an $(n+1)$-c.e. function that is not graph-n-c.e.

(c) There is a graph-2-c.e. function that is not ω-c.e.

(d) There is a 2-c.e. bijection f such that f^{-1} is not ω-c.e.

Cenzer, LaForte, and Remmel established the following results about characters in the Ershov hierarchy.

(i) For any n-c.e. character K, there is a computable equivalence structure with character K and without infinite equivalence classes.

(ii) There is an ω-c.e. character K such that any equivalence structure with character K must have infinite equivalence classes.

(iii) For any Δ_2^0 character K, there exists a d.c.e. equivalence structure with no infinite equivalence classes and with character K.

Cenzer, LaForte, and Remmel proved the following results about isomorphisms between equivalence structures in the Ershov hierarchy.

(I) For every $n \in \omega$, there exist two computable equivalence structures that are $(n+1)$-c.e. isomorphic, but not weakly n-c.e. isomorphic.

(II) There are two computable equivalence structures that are graph-2-c.e. isomorphic, but not weakly ω-c.e. isomorphic.

Cenzer, LaForte, and Remmel [3] also proved that a computable equivalence structure is computably categorical if and only if it is weakly ω-c.e. categorical. Furthermore, they showed that any computable equivalence structure with bounded character is relatively graph-2-c.e. categorical and that any computable equivalence structure with a finite number of infinite equivalence classes is relatively graph-ω-c.e. categorical. It then follows that a computable equivalence structure is Δ^0_2 categorical if and only if it is graph-ω-c.e. categorical.

We will prove a number of results about the complexity of isomorphisms of Σ^0_1 and of Π^0_1 equivalence structures. For example, in Section 2, we show that any Σ^0_1 equivalence structure $\mathcal{A}$ with infinitely many infinite equivalence classes is isomorphic to a computable structure. On the other hand, there are Σ^0_1 equivalence structures with finitely many infinite equivalence classes, which are *not* isomorphic to any computable structure. We show that if Σ^0_1 equivalence structures $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic to a computable structure $\mathcal{A}$ that is computably categorical or relatively Δ^0_2 categorical, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are Δ^0_2 isomorphic. In Section 3, we first observe that if $\mathcal{B}$ is a computably categorical computable equivalence structure any $\mathcal{A}$ is a Π^0_1 equivalence structure which is isomorphic to $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic. If $\mathcal{B}$ is a computable equivalence structure which is *not* computably categorical, then in several cases we construct a Π^0_1 structure $\mathcal{A}$ which is isomorphic to $\mathcal{B}$ but is *not* Δ^0_2 isomorphic to $\mathcal{B}$. The simplest case is when $\mathcal{B}$ consists of infinitely many equivalence classes of sizes 1 or 2, and no other classes; if $\mathcal{B}$ is Δ^0_2 categorical, then we show that the Π^0_1 structure $\mathcal{A}$ is moreover not Δ^0_2 isomorphic to any Σ^0_1 structure. In Section 4, we consider the *spectrum question*, which is to determine the possible sets (or degrees of sets) that can be the sets of elements in equivalence classes of size k, for some fixed k, in a computable equivalence structure of a given isomorphism type. For example, we show that for any infinite c.e. set B, there is a computable equivalence structure with infinitely many equivalence classes of size 1, infinitely many classes of size 2, and no other equivalence classes, such that $B = \{x : card([x]) = 2\}$. In Section 5, we consider the complexity of the theory $Th(\mathcal{A})$ of a computable equivalence structure $\mathcal{A}$, as well as the complexity of its elementary diagram $FTh(\mathcal{A})$. We explore the connection between the complexity of the character $\chi(\mathcal{A})$ and the theory $Th(\mathcal{A})$. We show that if $Th(\mathcal{A})$ is decidable, then the character $\chi(\mathcal{A})$ is computable. We show that if an equivalence structure $\mathcal{B}$ has a computable character, then there is a decidable structure $\mathcal{A}$ isomorphic to $\mathcal{B}$.

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2 Σ_1^0 equivalence structures

In this section, we consider properties of Σ_1^0 equivalence structures and their existence and categoricity. It is easy to show that the complexity of the character for Σ_1^0 equivalence structures is at the same level of the arithmetical hierarchy as for computable equivalence structures.

Lemma 2.1. *For any Σ_1^0 equivalence structure $\mathcal{A}$, we have:*
(a) $\{(k,a) : card([a]^{\mathcal{A}}) \geq k\}$ is a Σ_1^0 set;
(b) $Inf^{\mathcal{A}}$ is a Π_2^0 set;
(c) $\chi(\mathcal{A})$ is a Σ_2^0 set.

Thus, if $\mathcal{A}$ is a Σ_1^0 equivalence structure with infinitely many infinite equivalence classes, then it follows from Lemma 2.1 and Lemma 2.3 of [2] that $\mathcal{A}$ is isomorphic to a computable equivalence structure. However, it was shown in [2] that there is a Δ_2^0 character K such that any computable equivalence structure with character K must have infinitely many infinite equivalence classes. It was shown in [3] that for *any* Δ_2^0 character K, there is a d.c.e. equivalence structure $\mathcal{A}$ with character K and with no infinite equivalence classes. Hence there is a d.c.e. $\mathcal{A}$ that is not isomorphic to any computable equivalence structure. Now, for Σ_1^0 equivalence structures we have the following existence result.

Theorem 2.2. *For any Σ_2^0 character K and any finite $m \geq 1$, there is a Σ_1^0 equivalence structure $\mathcal{A}$ with character K and with exactly m infinite equivalence classes.*

Proof. Let K be a Σ_2^0 character. Let $\mathcal{B}$ be the equivalence structure given by Lemma 2.3 in [2] such that $\mathcal{B}$ has character K and infinitely many infinite equivalence classes, and, in addition, $Fin^{\mathcal{B}}$ is a Π_1^0 set. Simply define $\mathcal{A} = (\omega, E^{\mathcal{A}})$ by $E^{\mathcal{A}} = E^{\mathcal{B}} \cup (Inf^{\mathcal{B}} \times Inf^{\mathcal{B}})$. Then the structure $\mathcal{A}$ is Σ_1^0 since $Inf^{\mathcal{B}}$ is a Σ_1^0 set, $\mathcal{A}$ has the same character K, and the infinitely many infinite equivalence classes of $\mathcal{B}$ collapse into a single equivalence class $Inf^{\mathcal{A}}$ in $\mathcal{A}$. For $m > 1$, we can then append $(m-1)$ computable infinite equivalence classes. $\square$

Corollary 2.3. *There exists a Σ^0_1 equivalence structure $\mathcal{A}$ that is not isomorphic to any computable equivalence structure.*

Proof. Let K be a Σ^0_2 character that does not have an s_1-function. Then, by Lemma 2.6 of [2], there is no computable structure with character K and with finitely many infinite equivalence classes. □

We will next consider the effective categoricity of Σ^0_1 equivalence structures. It was shown in [2] that a computable equivalence structure $\mathcal{A}$ is computably categorical if and only if $\mathcal{A}$ is relatively computably categorical, and that a computable equivalence structure $\mathcal{A}$ is computably categorical if and only if one of the following conditions is satisfied:

1. $\mathcal{A}$ has only finitely many finite equivalence classes, or
2. $\mathcal{A}$ has finitely many infinite equivalence classes and bounded character, and there is at most one finite k such that $\mathcal{A}$ has infinitely many equivalence classes of size k.

It is also shown in [2] that a computable equivalence structure $\mathcal{A}$ is relatively Δ^0_2 categorical if and only if $\mathcal{A}$ has finitely many infinite equivalence classes or $\mathcal{A}$ has a bounded character.

Clearly, a noncomputable Σ^0_1 structure cannot be computably isomorphic to a computable structure, but we have the following best possible result.

Theorem 2.4. *Let $\mathcal{A}$ be a Σ^0_1 equivalence structure. Let $\mathcal{B}$ be a computable equivalence structure isomorphic to $\mathcal{A}$ such that $\mathcal{B}$ is computably categorical or relatively Δ^0_2 categorical. Then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic.*

Proof. Suppose first that $\mathcal{B}$ is computably categorical. It follows from Theorem 3.16 of [2] that $\mathcal{B}$ is relatively computably categorical. Hence there is an isomorphism f from $\mathcal{B}$ and $\mathcal{A}$, which is computable in $\mathcal{A}$. Since $\mathcal{A}$ is Σ^0_1, it follows that f is Δ^0_2.

Next, suppose that $\mathcal{B}$ is relatively Δ^0_2 categorical. Then:
(i) $\mathcal{B}$ has finitely many infinite equivalence classes, or
(ii) $\mathcal{B}$ has bounded character.

First, consider the computable structure $\mathcal{B}$. In Case (i), it is immediate that both $Inf^{\mathcal{B}}$ and $Fin^{\mathcal{B}}$ are computable. In Case (ii), it follows from Lemma 2.4 of [2] that there is a computable structure $\mathcal{B}'$ isomorphic $\mathcal{B}$ such that $Inf^{\mathcal{B}'}$ is computable. Thus, we may assume, without loss of generality, that $Inf^{\mathcal{B}}$ and $Fin^{\mathcal{B}}$ are computable.

Next, consider the Σ^0_1 structure $\mathcal{A}$. In Case (i), $Inf^{\mathcal{A}}$ is Σ^0_1 since there is a finite set of representatives $\{a_1, \ldots, a_m\}$ for the infinite classes, so $a \in Inf^{\mathcal{A}} \iff aE^{\mathcal{A}}a_1 \vee \cdots \vee aE^{\mathcal{A}}a_m$. In Case (ii), $Inf^{\mathcal{B}}$ is also Σ^0_1. That is, if n is an upper bound for the size of a finite equivalence class, then $a \in Inf^{\mathcal{B}} \iff card([a]) > n$.

Thus, both $Fin^{\mathcal{A}}$ and $Inf^{\mathcal{A}}$, and $Fin^{\mathcal{B}}$ and $Inf^{\mathcal{B}}$ are computable in $\emptyset'$. Moreover, it is easy to see that if $x \in Fin^{\mathcal{A}}$, then we can find the equivalence class $[x]^{\mathcal{A}}$ computably in $\emptyset'$. That is, we simply search until we find an n such that $\{y : yE^{\mathcal{A}}x \;\&\; y > n\}$ is empty, which we can decide from an $\emptyset'$-oracle. Then we know that $[x]^{\mathcal{A}} = \{z : zE^{\mathcal{A}}x \;\&\; z \leq n\}$, which also can be computed from an $\emptyset'$-oracle. Similarly, we can find the equivalence class $[y]^{\mathcal{B}}$ computably in $\emptyset'$ for any $y \in Fin^{\mathcal{B}}$. Then we can use a simple back-and-forth argument to define an isomorphism $f : Fin^{\mathcal{A}} \to Fin^{\mathcal{B}}$ that is computable in in $\emptyset'$. That is, computably in $\emptyset'$, we can compute enumerations $a_0 < a_1 < \cdots$ of $Fin^{\mathcal{A}}$, and $b_0 < b_1 < \cdots$ of $Fin^{\mathcal{B}}$. We then define f in stages, and let f_s denote the finite function defined at the end of stage s.

Stage 0. Search for the least b_i such that $card([a_0]^{\mathcal{A}}) = card([b_i]^{\mathcal{B}})$ and define f_0 so that it maps $[a_0]^{\mathcal{A}}$ onto $[b_i]^{\mathcal{B}}$ in an increasing fashion. If $i > 0$, then we search for the least a_j such that $card([a_j]^{\mathcal{A}}) = card([b_0]^{\mathcal{B}})$, and then define f_0 so that it maps $[a_j]^{\mathcal{A}}$ onto $[b_0]^{\mathcal{B}}$ in an increasing fashion.

Stage $s+1$. Assume we have defined f_s so that its domain and range are finite unions of equivalence classes in $\mathcal{A}$ and $\mathcal{B}$, respectively, and that $\bigcup_{i=0}^{s}[a_i]^{\mathcal{A}}$ is contained in the domain of f_s, and $\bigcup_{i=0}^{s}[b_i]^{\mathcal{B}}$ is contained in the range of f_s. Then, to extend f_s to f_{s+1}, we search for the least i such that a_i is not in the domain of f_s. We then search for the least b_k not in the range of f_s such that $card([a_i]^{\mathcal{A}}) = card([b_k]^{\mathcal{B}})$, and define f_{s+1} so that it maps $[a_i]^{\mathcal{A}}$ onto $[b_k]^{\mathcal{B}}$ in an increasing fashion. Next, we search for the least n such that b_n is not in the range of f_s and $n \neq k$. We then search for the least m such that a_m is not in the domain of f_s, and $m \neq i$, and $card([a_m]^{\mathcal{A}}) = card([b_n]^{\mathcal{B}})$. Then define f_{s+1} so that it maps $[a_m]^{\mathcal{A}}$ onto $[b_n]^{\mathcal{B}}$ in an increasing fashion.

It is easy to see that the construction is computable in $\emptyset'$, and that at each stage we can find the appropriate elements since we are assuming that $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$ are isomorphic. Thus, f will be a Δ^0_2 function, which is an isomorphism from $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$.

Similarly, it is easy to construct a function g that is computable in $\emptyset'$ and which is an isomorphism from $Inf^{\mathcal{A}}$ and $Inf^{\mathcal{B}}$. Hence $\mathcal{A}$ is Δ_2^0 isomorphic to $\mathcal{B}$. □

Corollary 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic Σ_1^0 equivalence structures that satisfy one of the following conditions:*

(i) *$\mathcal{A}$ has bounded character, or*

(ii) *$\mathcal{A}$ has only finitely many infinite equivalence classes.*

Then $\mathcal{A}$ and $\mathcal{B}$ are Δ_2^0 isomorphic.

Next, we briefly discuss the connection with bi-reducibility of equivalence structures as studied by Bernardi and Sorbi [1], Lachlan [10], and Gao and Gerdes [6].

We say that one equivalence relation R is *strongly reducible* to another equivalence relation S (written $R \leq S$) if and only if there exists a computable function f such that for all $x, y \in \omega$,

$$xRy \Leftrightarrow f(x)Sf(y).$$

If $R \leq S$ and $S \leq R$, then R and S are *bi-reducible.* If (ω, R) is computably isomorphic to (ω, S), then they are certainly bi-reducible. The converse does not hold. That is, Gerdes and Gao [6] proved that every computable equivalence relation is bi-reducible to one of the following types:

1. for some finite n, the equivalence relation $x \equiv y \ mod \ n$, which defines a computable equivalence structure with n infinite equivalence classes and without finite classes;

2. the equality relation, which defines a computable equivalence structure with infinitely many classes of size one, and no other classes.

Thus, the partial ordering $(C, <)$ of the computable equivalence structures modulo strong reducibility, is isomorphic to $\omega + 1$. In fact, it is easy to see that two computable equivalence structures are bi-reducible if and only if they have the same number of equivalence classes. Thus, in particular, bi-reducible structures need not be isomorphic. For example, if $\mathcal{A}$ consists of infinitely many classes of size 1, and $\mathcal{B}$ consists of infinitely many classes of size 2, then $\mathcal{A}$ and $\mathcal{B}$ are bi-reducible but not isomorphic. Furthermore, we have already seen that, even if two computable equivalence structures are isomorphic, they need not be computably isomorphic. For computable equivalence structures, the given effective notion of bi-reducibility is identical to the noneffective version.

A Σ^0_1 equivalence relation (or ceer) S is said to be *universal* if $R \leq S$ for any Σ^0_1 equivalence relation R. Bernardi and Sorbi [1] showed that universal ceers exist.

3 Π^0_1 equivalence structures

In this section, we show that even simple Π^0_1 equivalence structures do not have to be Δ^0_2 isomorphic to computable structures. Note that if $\mathcal{B}$ is a Π^0_1 equivalence structure, and $\mathcal{A}$ is an isomorphic computable structure that is computably categorical, then, since $\mathcal{A}$ is also relatively computably categorical, $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic. Thus, we have the following result.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic Π^0_1 equivalence structures such that $\mathcal{A}$ satisfies one of the following conditions:*

(i) *$\mathcal{A}$ has only finitely many finite equivalence classes, or*

(ii) *$\mathcal{A}$ has finitely many infinite equivalence classes and bounded character, and there is at most one finite k such that $\mathcal{A}$ has infinitely many equivalence classes of size k.*

Then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic.

However, our next two results show that Theorem 3.1 does not extend to all equivalence structures that are isomorphic to computable, relatively Δ^0_2 categorical structures.

Theorem 3.2. *Suppose that $\mathcal{B}$ is a computable equivalence structure with a bounded character for which there exist $k_1 < k_2 \leq \omega$ such that $\mathcal{B}$ has infinitely many equivalence classes of size k_1 and infinitely many equivalence classes of size k_2. Then there exists a Π^0_1 structure $\mathcal{A}$ that is not Δ^0_2 isomorphic to $\mathcal{B}$ and, moreover, $\mathcal{A}$ is not Δ^0_2 isomorphic to any Σ^0_1 structure.*

Proof. We first suppose that $\mathcal{B}$ has no other equivalence classes. It suffices to build a Π^0_1 equivalence structure $\mathcal{A}$ such that $\{a : card([a]^{\mathcal{A}}) = k_2\}$ is not a Δ^0_2 set. That is, it follows from Lemma 2.1 that for any Σ^0_1 structure, the set of elements that belong to an equivalence class of (finite) size k is a Δ^0_2 set. So if $\mathcal{A}$ were Δ^0_2 isomorphic to a Σ^0_1 structure, then $\mathcal{A}$ would also have this property.

For simplicity of the construction, we let $\mathcal{A}$ have universe $\omega \setminus \{0\}$. Let $\phi : \omega^3 \to \{0, 1\}$ be a computable function such that for every Δ^0_2 set D, there is some e for which for all $n \in \omega$, the limit $\delta_e(n) =_{def} \lim_{t\to\infty} \phi(t, e, n)$ exists and δ_e is the characteristic function of D. If $\delta_e(n)$ is defined and has values in $\{0, 1\}$ for all n, we let $D_e = \{n : \delta_e(n) = 1\}$. The function ϕ exists by the Limit Lemma (see [14]). We will construct the equivalence relation $E = E^{\mathcal{A}}$ so that for each e, if D_e exists, then $card([2^e]^{\mathcal{A}}) = k_1$ if and only if $2^e \notin D_e$.

We construct $E^{\mathcal{A}}$ in stages. That is, at each stage s, we shall define a computable equivalence relation E_s so that $E_{s+1} \subseteq E_s$ for all s, and $E^{\mathcal{A}} = \bigcap_s E_s$. Let $[a]_s$ denote the equivalence class of a in E_s. At each stage s, we shall also define an *intended* equivalence class $I_s[2^e]$, either of size k_1 or of size k_2. We will ensure that for each e, there is some stage s_e such that for all $s \geq s_e$, we have $[2^e] = I_s[2^e]$. Furthermore, for all s, $[2^e]_{s+1} \subseteq [2^e]_s$, and $\bigcap_s [2^e]_s = [2^e]$. We shall also define a number of *permanent* classes $[a]$ of size k_1 at each stage s.

Construction

Stage 0. We start with the equivalence classes $\{2^e(2k+1) : k \in \omega\}$ for $e \geq 0$. For each $e \geq 0$, we let $I_0[2^e] = \{2^e, 3 \cdot 2^e, 5 \cdot 2^e, \ldots, (2k_1 - 1) \cdot 2^e\}$.

Stage s+1. At the end of stage s, assume that for each e, $[2^e]_s = \{2^e, a_1, a_2, \ldots\}$, and we have defined the intended equivalence class $I_s[2^e]$ so that $I_s[2^e]$ is an initial subset of $[2^e]_s$ with cardinality either k_1 or k_2. Moreover, assume that if $\phi(s, e, 2^e) = 1$, then $I_s[2^e]$ has cardinality k_1, and if $\phi(s, e, 2^e) = 0$, then $I_s[2^e]$ has cardinality k_2. For each e, we say that the element 2^e *requires attention* at stage $s+1$ if $\phi(s+1, e, 2^e) \neq \phi(s, e, 2^e)$.

If 2^e requires attention at stage $s+1$, we take the following action according to whether $I_s[2^e]$ has cardinality k_1 or k_2.

Case (i). If $card(I_s[2^e]) = k_2$, then let $I_{s+1}[2^e] = \{2^e, a_1, \dots, a_{k_1-1}\}$, let $[2^e]_{s+1} = \{2^e, a_1, \dots, a_{k_1-1}, a_{2k_1}, a_{2k_1+1}, \dots\}$, and create a permanent equivalence class $\{a_{k_1}, a_{k_1+2}, \dots, a_{2k_1-1}\}$ of size k_1.

Case (ii). If $card(I_s[2^e]) = k_1$, then do the following. First suppose that k_2 is finite. Then we let $I_{s+1}[2^e] = \{2^e, a_1, \dots, a_{k_2-1}\}$, let $[2^e]_{s+1} = \{2^e, a_1, \dots, a_{k_2-1}, a_{k_2+k_1}, a_{k_2+k_1+1}, \dots\}$, and create a permanent equivalence class $\{a_{k_2}, a_{k_2+1}, \dots, a_{k_2+k_1-1}\}$ of size k_1. If $k_2 = \omega$, then we simply let $I_{s+1}[2^e] = [2^e]_{s+1} = [2^e]_s$.

If 2^e does not require attention, then, again, there are two cases. If $k_2 = \omega$ and $I_s[2^e] = [2^e]_s$ is infinite, then we let $I_{s+1}[2^e] = [2^e]_{s+1} = [2^e]_s$. If $card([I_s[2^e]) = k_m$ is finite, then we let $I_{s+1}[2^e] = \{2^e, a_1, \dots, a_{k_m-1}\}$, let $[2^e]_{s+1} = \{2^e, a_1, \dots, a_{k_m-1}, a_{k_m+k_1}, a_{k_m+k_1+1}, \dots,$ and create a permanent equivalence class $\{a_{k_m}, a_{k_m+1}, \dots, a_{k_m+k_1-1}\}$ of size k_1.

Clearly, the equivalence relation E_s is uniformly computable, and we have $E_{s+1} \subseteq E_s$ for every s. Thus, $E = \bigcap_s E_s$ is a Π^0_1 equivalence relation.

First, we show that every equivalence class in E has either k_1 or k_2 elements. The elements which are (ever) removed from $[2^e]_s$ form permanent equivalence classes of size k_1. Thus, we only need to check the classes $[2^e]_s$ for each e. By our construction, $[2^e]_s$ is infinite for every s. There are two cases. If $\lim_{s\to\infty} \phi(s, e, 2^e)$ exists, then there is some stage s such that $\phi(s, e, 2^e) = \phi(t, e, 2^e)$ for all $t \geq s$. Let $[2^e]_s = I_s[2^e] \cup \{a_1 < a_2 < \cdots\}$. If $I_s[2^e]$ has cardinality k_2 and $k_2 = \omega$, then $[2^e]_t = [2^e]_{t+1}$ for all $t \geq s$ so that $[2^e]$ is infinite. If $I_s[2^e]$ has cardinality k_1 or k_2, and k_2 is finite, then $[2^e]_{s+n} = I_s[2^e] \cup \{a_{k_n+1} < a_{k_n+2} < \cdots\}$, so $[2^e] = I_s[2^e]$ which, by construction, has cardinality either k_1 or k_2.

Next, suppose that there are infinitely many s such that $\phi(s+1, e, 2^e) \neq \phi(s, e, 2^e)$. Let $s_0 < s_1 < \cdots$ be the stages $s+1$ such that $\phi(s, e, 2^e) = 1$ and $\phi(s+1, e, 2^e) = 0$ so that $card(I_s[2^e]) = k_2$ and $card(I_{s+1}[2^e]) = k_1$. At each such stage s_n, we will remove the second k_1 elements from $[2^e]_{s_n}$ and make it a permanent equivalence class of size k_1. Thus, it follows that $\cap_n [2^e]_{s_n} = \{2^e, 3 \cdot 2^e, \dots, (2k_1 - 1) \cdot 2^e\}$, so that $card([2^e]) = k_1$.

Next, we check that $A = \{e : card([2^e]) = k_2\}$ is not a Δ^0_2 set. If it were, then, for some e, $\chi_A(n) = \lim_{s\to\infty} \phi(s, e, n)$ exists for all n. Let t_0 be large enough so that $\chi_A(2^e) = \phi(s, e, 2^e)$ for all $s \geq t_0$. By the construction, $card([2^e]_{t_0}) = k_2$ if and only if $\phi(t_0, e, 2^e) = 0$. By the definition of t_0, $\chi_A(2^e) = \phi(s, e, 2^e)$ and the element 2^e never requires attention after stage t_0, so that $card([2^e]) = card([I_{t_0}[2^e]$. Thus, $card([2^e]) = k_2$ if and only if $\lim_{s\to\infty} \phi(s, e, 2^e) \neq \chi_A(2^e)$.

Finally, suppose that $\mathcal{B}$ is a structure that has bounded character and has infinitely many classes of size k_1 and of size k_2, but also has other equivalence classes. Recall that the character $\chi(\mathcal{B})$ is a Σ_2^0 set. Now we may remove $\{(k_i, n) : n \in \omega \;\&\; i \in \{1, 2\} \;\&\; k_i \text{ is finite}\}$ from $\chi(\mathcal{B})$ and still have a Σ_2^0 character K. We now have several cases.

First, suppose that k_1 is finite and $k_2 = \omega$. Then K is a bounded character, and hence we can construct a computable equivalence structure $\mathcal{C}$ with character K. Let $\mathcal{A}$ be the Π_1^0 structure that has infinitely many equivalence classes of size k_1 and k_2, but no other equivalence classes, and which is not Δ_2^0 isomorphic to any Σ_1^0 structure. Then the disjoint union $\mathcal{A} \oplus \mathcal{C}$ will be isomorphic to $\mathcal{B}$. We may assume that k_1 is the largest size of an equivalence class that is finite and such that there are infinitely many equivalence classes of that size in $\mathcal{B}$. Thus, in any Σ_1^0 structure $\mathcal{B}^*$ that is isomorphic to $\mathcal{B}$, the set S of all elements that belong to finite equivalence classes of sizes bigger than k_1 is finite. The set D of elements d in $\mathcal{B}^*$ such that $card([d]^{\mathcal{B}^*}) \geq k_1$ is clearly a c.e. set, so that $D - S$ is a c.e. set consisting of all elements of $\mathcal{B}^*$ the equivalence classes of which are of sizes k_1 or k_2. Then, clearly, $\mathcal{B}^* \upharpoonright D - S$ is computably isomorphic to a Σ_1^0 structure. But then $\mathcal{A} \oplus \mathcal{C}$ cannot be Δ_2^0 isomorphic to any such Σ_1^0 structure $\mathcal{B}^*$, since any isomorphism would have to map $\mathcal{A}$ onto $\mathcal{B}^* \upharpoonright (D - S)$.

Next, suppose that k_1 and k_2 are finite and $\mathcal{B}$ has $r < \omega$ infinite equivalence classes. It is easy to modify the construction to ensure that $\mathcal{A}$ has r infinite equivalence classes, in addition to infinitely many equivalence classes of size k_1 and infinitely many equivalence classes of size k_2, so that $\mathcal{A}$ is not Δ_2^0 isomorphic to any Σ_1^0 structure. Since K is a bounded character, it is easy to construct a computable structure $\mathcal{C}$ with character K and no infinite equivalence classes. Thus, $\mathcal{A} \oplus \mathcal{C}$ will be a Π_1^0 structure which is isomorphic to $\mathcal{B}$. We may assume that k_1 and k_2 are the two largest sizes of equivalence classes that are finite and such that there are infinitely many equivalence classes of those sizes in $\mathcal{B}$. Thus, in any Σ_1^0 structure $\mathcal{B}^*$ isomorphic to $\mathcal{B}$, there are only finitely many elements S that belong to finite equivalence classes the sizes of which are bigger than k_2. The set D of elements d in $\mathcal{B}^*$

such that $card([d]^{\mathcal{B}^*}) \geq k_1$ is clearly a c.e. set, so that $D - S$ is a c.e. set consisting of all elements of $\mathcal{B}^*$ the equivalence classes of which are of sizes k_1 or k_2, together with r infinite equivalence classes in $\mathcal{B}^*$. Clearly, $\mathcal{B}^* \upharpoonright D$ is computably isomorphic to a Σ^0_1 structure. However, then $\mathcal{A} \oplus \mathcal{C}$ cannot be Δ^0_2 isomorphic to any such Σ^0_1 structure $\mathcal{B}^*$, since such an isomorphism would have to map $\mathcal{A}$ onto $\mathcal{B}^* \upharpoonright (D - S)$. □

Corollary 3.3. *If $\mathcal{B}$ is a computable equivalence structure with bounded character which is not computably categorical, then there exists a Π^0_1 structure $\mathcal{A}$ that is not Δ^0_2 isomorphic to $\mathcal{B}$, and, moreover, $\mathcal{A}$ is not Δ^0_2 isomorphic to any Σ^0_1 structure.*

Proof. Let the computable equivalence structure $\mathcal{B}$ have bounded character such that $\mathcal{B}$ is not computably categorical. Suppose first that $\mathcal{B}$ has only finitely many infinite equivalence classes. It was proved in [2] that if $\mathcal{B}$ has finitely many infinite equivalence classes, and at most one finite k such that there are infinitely many equivalence classes of size k, then $\mathcal{B}$ is computably categorical (see Corollary 3.3 and Theorem 3.16 of [2]). Hence there exist finite $k_1 < k_2$ such that $\mathcal{B}$ has infinitely many equivalence classes of size k_1 and infinitely many equivalence classes of size k_2. Next, suppose that $\mathcal{B}$ has infinitely many infinite equivalence classes. If $\mathcal{B}$ has a finite character, then $\mathcal{B}$ is computably categorical. Thus, $\chi(\mathcal{B})$ is both bounded and infinite, so that there must exist a finite k such that $\mathcal{B}$ has infinitely many equivalence classes of size k, as well as infinitely many infinite equivalence classes. Thus, Theorem 3.2 applies in either case. □

Next, we shall consider structures with unbounded characters and with only finitely many infinite equivalence classes.

Theorem 3.4. *Suppose that $\mathcal{B}$ is a computable equivalence structure that has an unbounded character and only finitely many infinite equivalence classes (and is therefore relatively Δ^0_2 categorical). Then there exists a Π^0_1 structure $\mathcal{A}$ that is isomorphic to $\mathcal{B}$, but not Δ^0_2 isomorphic to $\mathcal{B}$, and, moreover, $\mathcal{A}$ is not Δ^0_2 isomorphic to any Σ^0_1 structure.*

Proof. Let $\phi : \omega^3 \to \{0, 1\}$ be the computable function defined in Theorem 3.2 for which for every Δ^0_2 set D, there is some e such that for all $n \in \omega$, the limit $\delta_e(n) =_{def} \lim_{t\to\infty} \phi(t, e, n)$ exists and δ_e is the characteristic function of D.

By Lemma 2.6 of [2], there is a computable s_1-function f such that $m_i = lim_s f(i,s)$ exists and is finite for all i, and such that for each i, $\mathcal{B}$ has an equivalence class of size m_i. Note that $M = \{m_i : i \in \omega\}$ is a Δ^0_2 set. Thus, by Lemma 2.8 of [2], there exists a computable structure which consists of exactly one equivalence class of size m_i for each i.

First, assume that $\mathcal{B}$ has no other equivalence classes, i.e., $\mathcal{B}$ consists of exactly one equivalence class of size m_i for each i. It suffices to build a Π^0_1 equivalence structure $\mathcal{A}$ such that $\{a : card([a]^{\mathcal{A}}) = m_{2i} \text{ for some } i\}$ is not a Δ^0_2 set. That is, we observe that the functions f_E and f_O, defined by $f_E(i,s) = f(2i,s)$ and $f_O(i,s) = f(2i+1,s)$ are also s_1-functions so it follows by Lemma 2.7 of [2] that the sets $M_0 = \{m_{2i} : i \in \omega\}$ and $M_1 = \{m_{2i+1} : i \in \omega\}$ are both Δ^0_2 and hence there exist computable structures $\mathcal{B}_0$ and $\mathcal{B}_1$ which consist of precisely one class of size m_{2i} for B_0 (respectively m_{2i+1} for B_1). Hence in the structure $B_0 \oplus B_1$, the set $\{x : card([x]) \in M_0\}$ is computable. Since we have assumed that $\mathcal{B}$ is relatively Δ^0_2 categorical, it follows from Theorem 2.4 that for any Σ^0_1 equivalence structure with character $\{\langle m,1\rangle : m \in M_0 \cup M_1\}$, the set $\{x : card([x]) \in M_0\}$ is Δ^0_2.

The construction of $E^{\mathcal{A}}$ is again by stages. That is, at each stage s we shall define a computable equivalence relation E_s so that $E_{s+1} \subseteq E_s$ for all s, and $E^{\mathcal{A}} = \bigcap_s E_s$. Again, we let $[a]_s$ denote the equivalence class of a in E_s, and we let $I_s[a]$ denote the *intended* equivalence class of a at stage s. At any given stage s, the intended classes have exactly the sizes $f(i,s)$ for $i \in \omega$ and $I_s[2^e]$ will either of size $f(2^e,s)$ or be of size $f(1+2^e(2j+1),s)$ for some j. The construction will ensure that, for each i, there exists t such that $f(i,t) = m_i$ and the class of size m_i has become permanent. For each e, 2^e belongs to the class of size $f(2^e,s)$ at stage s if and only if $\phi(s,e,2^e) = 0$. It follows that for each e, if D_e exists, then $card([2^e]^{\mathcal{A}}) \in M_1$ if and only $2^e \notin D_e$.

Construction

Stage 0: E_0 consists of the equivalence classes $\{2^n(2k+1) : k \in \omega\}$ for $n \geq 0$. For each $e \geq 0$, we let $I_0[2^e] = \{2^e, 3\cdot 2^e, 5\cdot 2^e, \ldots, (f(2^e,0)-1)\cdot 2^e\}$. We then partition the remaining elements of $\{2^n(2k+1) : k \in \omega\}$ consecutively into the intended classes $I(2^e(2k+1),0)$ of size $f(2^e(2k+1),0)$ for $k > 0$.

Stage $s+1$: There are three tasks to accomplish at stage $s+1$. We will perform them sequentially.

First, we suppose $f(i, s+1) > f(i, s)$ for some i. In fact, we can construct f such that this occurs for exactly one i and that, in fact, $f(i, s+1) = f(i, s) + 1$. Now the class $I_s[a]$ intended to have size m_i lies in some infinite class $[a]_s = C_0$, where it is followed by intended classes $C_1, C_2, \dots$. Here we assume that for any $0 \leq i < j$, the elements of C_i are all smaller than the elements of C_j. The required action is to take for each $i \geq 0$, the first element of C_{i+1} and move it to C_i. This will make $card(I_{s+1}[a]) = f(i, s+1)$, while leaving the other intended classes with the same cardinalities.

Second, we may have $\phi(s+1, e, 2^e) \neq \phi(s, e, 2^e)$. Again we assume this occurs for exactly one e. Here the class $I_s[2^e]$ is an initial subset of some infinite class $[2^e]_s$, beginning with 2^e, and is followed by intended classes $C_1, C_2, \dots$ having cardinalities $c_1, c_2, \dots$, respectively. Let $[2^e]_s \cup C_1 \cup C_2 \cup \dots = \{a_1, a_2, \dots\}$. Suppose that the previous requirements have changed the intended size from $f(2^e, r)$ or from $f(1 + 2^e(2i+1), r)$ for each $i < n$. The required action now is to change the cardinality of $I[2^e]$ either from $f(2^e, s)$ to $f(1 + 2^e(2n+1), s)$ or vice versa. Suppose that intended cardinality $I_{s+1}[2^e]$ is now going to be $c_0 = f(j, s)$ where $j \in \{2^e, 1 + 2^e(2n+1)\}$. Let $d_i = c_0 + \dots + c_i$ for each i. Then we let $I_{s+1}[2^e] = \{a_1, \dots, a_{c_0}\}$ and, for each i, we convert C_{i+1} into $\{a_{d_i}, a_{d_i+1}, \dots, a_{d_i+c_{i+1}-1}\}$, so that the other classes maintain their cardinality. Next, we declare that $I[2^e]_{s+1}$ is intended for m_j. Observe once again that for any $i < j$, the elements of C_i are all smaller than the elements of C_j.

Finally, suppose that $I[2^e]$ was previously intended to have size m_k and now has size m_j. Then we have to work on the class C which was previously intended to have size m_j and change it over to size $f(j, s+1)$. The intended class C lies in the middle of some infinite class and we proceed as we did for the class $[2^e]$ above. Again, we observe that the elements of the class C are all larger than those of the class $I_s[2^e]$.

Third, we have to ensure that the actual classes $[a]_s$ will converge to finite classes with the intended cardinality. We accomplish this as follows. For each class $C = [a]_s$, let C be the union of intended classes $C_1, C_2, \dots$ We partition C into new classes $D_i = \cup_k C_{(2k+1)\cdot 2^i}$. In this way we ensure that any two intended classes will eventually be separated.

This completes the construction.

We claim that for each a, the class $I[a]_s$ eventually converges to the class $[a]$ and is associated with some intended cardinality m_i.

First, consider the class $[2^e]$. Choose t such that $f(2^e,t) = m_{2^e}$. Suppose first that $\phi(s+1,e,2^e) \neq \phi(s,e,2^e)$ infinitely often. Then infinitely often we have $I_s[2^e] = C$ as the first m_{2^e} elements of $[2^e]_s$, and at the other type of stages we have $C \subseteq I_s[2^e]$, so that $C \subseteq [2^e]$. By the third type of action, all other elements are eventually not equivalent to 2^e, so that $[2^e] = C$ and $[2^e]$ has cardinality m_{2^e}. Next, suppose that $\phi(s+1,e,2^e) \neq \phi(s,e,2^e)$ only finitely many times. Then we may assume that after stage t, $I[2^e]$ is never changed by the second type of action.

There are two possibilities. If $I_t[2^e]$ is intended to have size $f(2^e,t) = m_{2^e}$, then it cannot be affected by the first type of action (by the assumption above), and hence it cannot be affected by the third type of action, since it does not change any of the intended classes. If $I_t[2^e]$ is intended to have size $f(1+2^e(2j+1),t)$ for some j, then the intended size of this class will not be changed again by any action of the first type. Hence, once $f(1+2^e(2j+1),s)$ stops changing, it will have a fixed size. Since $I_s[2^e]$ is always an initial segment of $[2^e]_s$ it is never affected by any other type of action. Thus, it will stabilize to a class of size m_k, where $k = 2^e(2j+1)$. Finally, actions of the third type will eventually remove all other elements from $[2^e]$.

Now, consider elements a which do not end up in $[2^e]$ for any e. It follows from the construction (by the third type of action) that $I_s[a]$ is eventually an initial segment of $[a]_s$. Take t large enough so that:

1. $a \notin [2^e]_s$ for any e and any $s > t$,
2. $I_s[a]$ is an initial segment of $[a]_s$ for any $s > t$,
3. $f(k,t) = m_k$, where $I_t[a]$ is intended to have size m_k.

Then for any $s > t$, $I[a]_s$ will be an initial segment of $C = [a]_s$ of size m_k, and hence $C \subseteq [a]$. By the third type of action, no other elements will belong to $[a]$, and thus $[a] = C$.

Finally, suppose that $D_e(x) = lim_s \phi(s,e,x)$ is a Δ^0_2 set. Then by the construction, $I[2^e]$ will stabilize once $\phi(s,e,x,)$ has stabilized and we will have $card([2^e]) \in M_1 \iff 2^e \notin D_e$.

Thus, in our Π^0_1 structure $\mathcal{A}$, $\{x : card([x]) \in M_0\}$ is not a Δ^0_2 set and therefore $\mathcal{A}$ is not Δ^0_2 isomorphic to any Σ^0_1 structure.

Now, suppose that $\mathcal{B}$ does not consist only of one equivalence class of size m_i for each $i \geq \omega$. Moreover, suppose that $\mathcal{B}$ has r infinite equivalence classes for some $r < \omega$. In this case, we will take the corresponding s_1-function f and let $N_0 = \{m_{3i} : i \in \omega\}$, $N_1 = \{m_{3i+1} : i \in \omega\}$, and $N_2 = \{m_{3i+2} : i \in \omega\}$. Then N_0, N_1, and N_2 are Δ^0_2 sets. The sets $K_0 = \{(k,n) \in K : k \in N_0\}$, $K_1 = \{(k,n) \in K : k \in N_1\}$, and $K_2 = \{(k,n) \in K : k \notin N_0 \cup N_1\}$, are Σ^0_2 characters. Each of these sets has an s_1-function since $g_0(i) = f(3i)$ is an s_1-function for K_0, $g_1(i) = f(3i+1)$ is an s_1-function for K_1, and $g_2(i) = f(3i+2)$ is an s_1-function for K_2. By Lemma 2.8 of [2], there exist computable equivalence structure $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ such that:

1. $\mathcal{R}$ has character K_0 an no infinite equivalence classes,
2. $\mathcal{S}$ has character K_1 an no infinite equivalence classes, and
3. $\mathcal{R}$ has character K_2 and r infinite equivalence classes.

Thus, $\mathcal{B}$ is isomorphic to $\mathcal{B}^* = \mathcal{R} \oplus (\mathcal{S} \oplus \mathcal{T})$. Clearly, in $\mathcal{B}^*$, $\{x : card([x]^{\mathcal{B}^*}) \in N_0\}$ is a computable set. Thus, by the relative Δ^0_2 categoricity of $\mathcal{B}$, it must be the case that in any Σ^0_1 structure $\mathcal{D}$ isomorphic to $\mathcal{B}$, the set $\{x : card([x]^{\mathcal{D}}) \in N_0\}$ is Δ^0_2.

However, we can clearly modify the construction so that we obtain a Π^0_1 equivalence structure $\mathcal{A}$ such that $\mathcal{A}$ has exactly one equivalence class of size m_{3i} and one equivalence class of size m_{3i+1} for all $i \in \omega$, and $\{x \in \mathcal{A} : card([x]^{\mathcal{A}}) \in N_0\}$ is not a Δ^0_2 set. Next, observe that, since $\mathcal{B}$ has a Σ^0_2 character K, the set $K^* = \{(k,n) \in K : k \notin N_0 \cup N_1\} \cup \{(k,n) : k \in N_0 \cup N_1 \ \&\ (k, n+1) \in K\}$ is also a Σ^0_2 character, which has an s_1-function witnessed by g_2. Thus, by Lemma 2.8 of [2], there is a computable structure $\mathcal{C}$ such that $\mathcal{C}$ has character K^* and r infinite equivalence classes. Hence $\mathcal{A} \oplus \mathcal{C}$ is a Π^0_1 structure that is isomorphic to $\mathcal{B}$. Now, if $\mathcal{A} \oplus \mathcal{C}$ were Δ^0_2 isomorphic to a Σ^0_1 structure $\mathcal{B}^*$, which is isomorphic to $\mathcal{B}$, then, since in $\mathcal{B}^*$, the set $\{x : card([x]^{\mathcal{B}^*}) \in N_0\}$ is a Δ^0_2 set, it would follow that $\{x : card([x]^{\mathcal{A} \oplus \mathcal{C}}) \in N_0\}$ is a Δ^0_2 set. However, if that were the case, then $\{2x : card([2x]^{\mathcal{A} \oplus \mathcal{C}}) \in N_0\}$ would also be a Δ^0_2 set, which it is not by the construction of $\mathcal{A}$. Thus, $\mathcal{A} \oplus \mathcal{C}$ cannot be Δ^0_2 isomorphic to a Σ^0_1 structure. □

By Corollary 4.8 of [2], a computable structure $\mathcal{A}$ is relatively Δ^0_2 categorical if and only if $\mathcal{A}$ has finitely many equivalence classes or $\mathcal{A}$ has a bounded character. Thus we can combine the previous two theorems to conclude the following.

Theorem 3.5. *Suppose that $\mathcal{B}$ is a computable equivalence structure that is relatively Δ^0_2 categorical, but not computably categorical. Then there exists a Π^0_1 structure $\mathcal{A}$ that is not Δ^0_2 isomorphic to $\mathcal{B}$ and, moreover, $\mathcal{A}$ is not Δ^0_2 isomorphic to any Σ^0_1 structure.*

Proof. If $\mathcal{B}$ has bounded character, this follows from Corollary 3.3. If $\mathcal{B}$ has unbounded character, then this follows from Theorem 3.4. □

We note that Theorem 3.5 does not cover all Δ^0_2 categorical computable equivalence structures since Kach and Turetsky [7] showed that there exists a computable Δ^0_2 categorical equivalence structure $\mathcal{B}$ which has infinitely many infinite equivalence classes and an unbounded character, but has no computable s_1-function, and has only finitely many equivalence classes of size k for any finite k. The next result will cover this case. In such a case, we shall show that there exists a Π^0_1 structure $\mathcal{A}$ which is isomorphic to $\mathcal{B}$ such that $Inf^{\mathcal{A}}$ is a Π^0_2 complete set.

Theorem 3.6. *Let $\mathcal{B}$ be a computable equivalence structure with infinitely many infinite equivalence classes and with unbounded character such that for each finite k, there are only finitely many equivalence classes of size k. Then there is a Π^0_1 structure $\mathcal{A}$ which is isomorphic to $\mathcal{B}$ such that $Inf^{\mathcal{A}}$ is Π^0_2 complete. Furthermore, if $\mathcal{B}$ is Δ^0_2 categorical, then $\mathcal{A}$ is not Δ^0_2 isomorphic to any computable structure.*

Proof. We fix a computable bijection $t : \omega^3 \to \omega$. For any subset $S \subseteq \omega^3$, the function t induces a total ordering of type ω on S by defining for $(a_1, b_1, c_1), (a_2, b_2, c_2) \in S$, $(a_1, b_1, c_1) < (a_2, b_2, c_2)$ if and only if $t(((a_1, b_1, c_1)) < t((a_2, b_2, c_2))$.

Let $\mathcal{B}$ have character K. Since K is Σ^0_2, there is a computable relation Q such that

$$(k, m) \in K \iff (\exists w)(\forall s)Q(s, w, k, m).$$

We may assume, without loss of generality, that for each (k, m), there is at most one w such that $(\forall s)C(s, w, k, m)$. Let $C = \{(w, k, m) : (\forall s)Q(s, w, k, m)\}$. Then there is a one-to-one correspondence between C and K given by mapping (w, k, m) to (k, m).

For each s, we have the uniformly computable set

$$C_s = \{(w, k, m) : \forall t < s)\ C(t, w, k, m)\},$$

which can be enumerated, relative to the order induced by t described above, as $\{(w,k,m)_{i,s} : i \in \omega\}$. For each s, $C_{s+1} \subseteq C_s$ and $\cap_s C_s = C$. For each k and m, we know that $(k, m+1) \in K \implies (k,m) \in K$, so that we may assume that if $(w,k,m) \notin C_s$, then for all v and all $n > m$, $(v,k,n) \notin C_s$.

We will construct the Π^0_1 equivalence relation E as the intersection $\cap_s E_s$ of uniformly computable equivalence relations defined at stage s. At stage s, we will have for each $(w,k,m) \in C_s$, some equivalence class that is intended to have cardinality k. For example, if $I[2e]$ is the intended equivalence class associated with (w,k,m) as stage s, then just as we did in the proof of Theorems 3.2 and 3.4, we will attempt to ensure that $[2e]$ has the correct cardinality by gradually removing all but the first k elements $[2e]_s$. That is, if at stage $t > s$, $[2e]_t = \{a_0 < a_1 < \cdots\}$, then we remove all elements of the form a_{k+2i} for $i \geq 0$ from $[2e]_t$ and have these elements form a new permanent infinite equivalence class.

Now, let P be a complete Π^0_2 set such that for some computable relation R we have for all n, $n \in P \iff \{x : R(n,x)\}$ is infinite). We may assume, without loss of generality, that for each x, there is exactly one n such that $R(n,x)$ and that $n \leq x$.

At each stage s of the construction, we will define an ordering $n_{0,s}, n_{1,s}, \ldots$ of ω of type ω so that the intended equivalence class $I_s[2n_{i,s}]$ is associated with $(w,k,m)_{i,s}$, which means that at stage s, we intend $[2n_{i,s}]$ to have cardinality k.

The construction will ensure that for each $i \notin P$, the class $[2i]$ eventually becomes associated with a fixed $(w,k,m) \in C$ and thus has finite cardinality k in $\mathcal{A}$. For each $i \in P$, the construction will ensure that $[2i]$ is associated with an increasing sequence of triples $(w,k,m)_s$ of larger and larger size so that in the limit, $[2i]$ is infinite. Thus, it will follow that $i \in P \iff 2i \in Inf^{\mathcal{A}}$, and, hence, $Inf^{\mathcal{A}}$ will be a Π^0_2 complete set.

Construction

Stage 0. At stage 0, we have C_0 enumerated as $(w,k,m)_{i,0} = (w_i, k_i, m_i) : i \in \omega\}$. For each i, we let $n_{i,0} = 2i$ for all i so that the intended class $I[2i]$ is to have cardinality k_i at stage 0. The odd numbers are partitioned among the classes $[2i]_0$ in some computable fashion, say $[2i]_0 = \{2i\} \cup \{1 + 2^{i+1}(2n+1) : n \in \omega\}$. Thus, $(x,y) \in E^0$ if and only if x and y belong to the same class $[2i]_0$ for some i.

Stage $s+1$. After stage s, we have an equivalence relation E^s and an ordering $2n_{1,s}, 2n_{2,s}, \dots$ of the even numbers, so that $2n_{i,s}$ is associated with the triple $(w,k,m)_{i,s}$. At stage $s+1$, let $i \leq s+1$ be the unique number such that $R(s+1, n_{i,s})$. Let $j > s+1$ be large enough so that no number $n_{p,s}$ with $p \geq j$ has been used during the construction. Then we simply move $n_{i,s}$ to location j and let all of the $n_{r,s}$ in between move down one position. That is, we let $n_{j,s+1} = n_{i,s}$, $n_{r,s+1} = n_{r+1,s}$ for all r with $i \leq r < j$ and $n_{r,s+1} = n_{r,s}$ for all r such that either $r < i$ or $r > j$. Finally, for all i, we define E^{s+1} as follows. For each class $[2n]$, let $n = n_{i,s+1}$, and let $(w,k,m) = (w,k,m)_{i,s+1}$. Suppose that $[2n]_s = \{2n < a_1 < a_2 < \cdots < a_{k-1} < a_k < \cdots\}$. Then we let $[2n]_{s+1} = \{2n, a_1, \dots, a_{k-1}, a_{k+1}, a_{k+3}, \dots\}$ and we create a new, permanent, infinite class $\{a_k, a_{k+2}, \dots\}$. Previously created permanent, infinite classes are left untouched.

Claim 1: If $n \notin P$, then there exist i and s such that for all $t \geq s$, $n = n_{i,t}$.

Proof of Claim 1. Since $\{x : R(n,x)\}$ is finite, we may choose t to be large enough so that for all $x > t$, $\neg R(n,x)$. Let $n = n_{i,t}$. It follows from the construction that for all $s > t$, if $n = n_{j,s}$, then $j \leq i$ and hence j can only decrease a finite number of times before becoming fixed at some stage s.

For $n \notin P$, let $i(n)$ be the limit of $\{i : n = n_{i,s}\}$, as shown to exist in Claim 1. Let $I = \{i(n) : n \notin P\}$ and denote n by N_i if $i = i(n)$.

Claim 2: $I = \omega$.

Proof of Claim 2. Observe that I is infinite since $\omega - P$ is infinite. Now suppose that $I \neq \omega$. Then there must be some i such that $i+1 \in I$ but $i \notin I$. Let $t > i$ be a stage such that $N_{i+1} = n_{i+1,s}$ for all $s \geq t$. Then for any $s > t$, it can never happen that $R(s+1, n_{i,s})$. Otherwise, the construction would make $n_{i,s+1} = N_{i+1}$, contrary to the choice of t.

Claim 3: For each $(k,m) \in K$, there exist at least m classes in $\mathcal{A}$ of size exactly k.

Proof of Claim 3. For each $(k,m) \in K$, we have some $(w,k,m) \in C$. After some stage s, we will have a fixed i such that $(w,k,m) = (w,k,m)_{i,t}$ for all $t > s$ and a fixed $n \notin P$ (by Claim 2) such that $n = N_i = n_{i,t}$ for all $t > s$. Suppose that $[2n]_s = \{2n, a_1, a_2, \dots\}$. Then, $I_t[2n] = \{2n, a_1, \dots, a_{k-1}\}$ for all $t > s$. It is easy to see that in such a situation $a_{k+r} \notin [2n]_{s+r+1}$ for all r. Hence $[2n] = \{2n, a_1, \dots, a_{k-1}\}$ and has size k as desired. Similarly for $1 \leq p < m$, we will have a class of size k and these classes will all be distinct.

Claim 4: If $n \in P$, then for any r, there exists s such that for all $t > s$, $n = n_{i,t}$ with $i > r$.

Proof of Claim 4. Given r, just let s be large enough so that for all $i \leq r$ and all $t > s$, we have $n_{i,t} = N_i$. Since each $N_i \notin P$, it follows that $n \neq n_{i,t}$ for any $i \leq r$.

Claim 5: If $n \in P$, then $[2n]$ is infinite.

Proof of Claim 5. Suppose $n \in P$. We will define an infinite sequence $\{a_1, a_2, \dots\}$ such that each a_i is in $[2n]$. Since there are only finitely many classes in $\mathcal{B}$ of any fixed finite size, there is only a finite number of elements in C of the form $(w, 1, r)$, say $(w^1, 1, 1), (w^2, 1, 2), \dots, (w^m, 1, m)$. Let r be large enough so that each of these elements is among the first r elements of C. Let s be large enough, by Claim 4, so that for all $t \geq s$, $n = n_{i,t}$ with $i > r$, and let the intended class be $I_s[2n] = \{2n, b_1, b_2, \dots\}$. It follows from the construction that $b_1 \in [2n]_t$ for all $t \geq s$, so we can define $a_1 = b_1$.

To determine b_{j+1}, we similarly find s_j large enough so that the intended class of $[2n]$ has at least $j+2$ elements for all $t \geq s_j$, and let b_{j+1} be the $j+2$-th element of $[2n]_{s_j}$.

It follows that $\mathcal{A}$ has infinitely many infinite classes.

Claim 6: $\chi(\mathcal{A}) = K$.

Proof of Claim 6. By Claim 3, we have $K \subseteq \chi(\mathcal{A})$. For the other direction, we have, by Claim 5, that $[2n]$ is infinite for $n \in P$. By Claim 1, we see that for $n \notin P$, there is some i such that $n = N_i$, and hence some (w, k, m) such that $I_s[2n]$ has size k for all sufficiently large s. So, by the construction, $[2n]$ is the unique class of size k corresponding to (k, m).

Finally, suppose that $\mathcal{B}$ is Δ^0_2 categorical. By Lemma 2.3 of [2], there is a computable equivalence structure $\mathcal{D}$ which has character K and infinitely many infinite equivalence classes such that $Fin^{\mathcal{D}}$ is a Π^0_1 set. Thus, $\mathcal{D}$ is isomorphic to $\mathcal{B}$ and $Inf^{\mathcal{D}}$ is Δ^0_2. Hence it must be the case that in any computable equivalence structure $\mathcal{C}$ that is isomorphic to $\mathcal{B}$, $Inf^{\mathcal{C}}$ must be Δ^0_2 and, hence, $\mathcal{A}$ is not Δ^0_2 isomorphic to $\mathcal{C}$. □

There is one final result in order to cover all possible computable equivalence structures.

Theorem 3.7. *Suppose that $\mathcal{B}$ is a computable equivalence structure with infinitely many infinite equivalence classes and with unbounded character and* *there is some finite k such that $\mathcal{B}$ has infinitely many equivalence classes of size k. Then there is a Π^0_1 structure $\mathcal{A}$ which is not Δ^0_2 isomorphic to $\mathcal{B}$ and, moreover, $\mathcal{A}$ is not Δ^0_2 isomorphic to any Σ^0_1 structure.*

Proof. By the proof of Theorem 3.2, there is a Π^0_1 structure $\mathcal{A}_0$ which consists of infinitely many infinite equivalence classes and infinitely many equivalence classes of size k such that $Fin(\mathcal{A}_0) = \{c : card([c]) = k\}$ is not Δ^0_2. Define the Σ^0_2 character K to be $\chi(\mathcal{B}) - k \times \omega$ and let $\mathcal{C}$ be a computable structure with character K. Now let $\mathcal{A} = A_0 \oplus C$, so that $\{a \in \mathcal{A} : card([a]) = k\} = \{a \in \mathcal{A}_0 : card([a]) = k\} \times \{0\}$ and is hence not a Δ^0_2 set. However, in *any* Σ^0_1 structure $\mathcal{D}$, the set $\{d \in \mathcal{D} : card([d]) = k\}$ is a Δ^0_2 set. Thus $\mathcal{A}$ is a Π^0_1 equivalence structure which is isomorphic to $\mathcal{B}$ but Δ^0_2 isomorphic to any Σ^0_1 structure. □

Thus we have the following.

Theorem 3.8. *Suppose that $\mathcal{B}$ is a computable equivalence structure which is not computably categorical. Then there is a Π^0_1 structure $\mathcal{A}$ which is isomorphic to $\mathcal{B}$ but is not Δ^0_2 isomorphic to $\mathcal{B}$.*

Proof. There are three possible cases for $\mathcal{B}$.

Case I: $\mathcal{B}$ has bounded character. Then since $\mathcal{B}$ is not computably categorical, $\mathcal{B}$ must have infinitely many infinite classes. Thus, there must exist a finite k_1 such that $\mathcal{B}$ has infinitely many classes of size k_1. Moreover, it must be the case that either $\mathcal{B}$ has infinitely many infinite equivalence classes or there exists a finite $k_2 \neq k_1$ such that $\mathcal{B}$ also has infinitely many equivalence classes of size k_2. Then the result follows from Theorem 3.2.

Case II: Suppose that $\mathcal{B}$ has unbounded character and has finitely many infinite equivalence classes. Then $\mathcal{B}$ is relatively Δ^0_2 categorical and the result follows from Theorem 3.4.

Case III: Suppose that $\mathcal{B}$ has unbounded character and infinitely many infinite equivalence classes. There are two possibilities. First, there may exist a finite k such that $\mathcal{B}$ has infinitely many equivalence classes of size k. Then the result follows from Theorem 3.7. Second, it may be that for each finite k, there are only finitely many classes of size k. Then the result follows from Theorem 3.6 if $\mathcal{B}$ is Δ^0_2 categorical and is easy if $\mathcal{B}$ is not Δ^0_2 categorical, since then there is a computable structure $\mathcal{A}$ which is not Δ_2^0 isomorphic to $\mathcal{B}$. □

For Δ_2^0 categorical structures, we have the following immediate corollary.

Corollary 3.9. *Suppose that $\mathcal{B}$ is a Δ_2^0 categorical, but not computably categorical. Then there is a Π_1^0 structure $\mathcal{A}$ that is isomorphic to $\mathcal{B}$ such that $\mathcal{A}$ is not Δ_2^0 isomorphic to* any *computable equivalence structure.*

4 Spectra of equivalence structures

In this section, we begin to examine the spectrum question for equivalence structures. For a computable (Σ_1^0, Π_1^0, respectively) equivalence structure $\mathcal{A}$ and any cardinal $k \leq \omega$, we consider the possible Turing degrees of $\{a : card([a]) = k\}$ and $\{a : card([a]) \geq k\}$. For example, we know that for any c.e. equivalence structure $\mathcal{A}$, $Inf^{\mathcal{A}}$ is Π_2^0 and $Fin^{\mathcal{A}}$ is Σ_2^0. Thus, a natural question is to ask whether there exists for any Σ_2^0 Turing degree $\mathbf{c}$, a computable equivalence structure $\mathcal{A}$ with $Fin^{\mathcal{A}}$ of degree $\mathbf{c}$. We will not pursue such question in this paper. Instead, we shall prove two results about spectra in computable structures.

We now give an initial result for computable equivalence structures with infinitely many equivalence classes of size 1, infinitely many equivalence classes of size 2, and with no other equivalence classes. Clearly, for such computable equivalence structure, the elements in classes of size 2 form a c.e. set, and the elements in classes of size 1 form a co-c.e. set. In this case, we obtain not only every c.e. degree, but also every c.e. set.

Theorem 4.1. *For any infinite c.e. set B, there is a computable equivalence structure $\mathcal{A}$ with character $\{1,2\} \times (\omega - \{0\})$ and no infinite equivalence classes such that $\{a : card([a]) = 2\} = B$.*

Proof. Let $\{b_0, b_1, \ldots\}$ be a computable 1-1 enumeration of B. We will first give an enumeration $\{c_0, c_1, \ldots\}$ of B such that for every n and each $i < 2n+1$, $c_i < c_{2n+1}$. Let $c_0 = b_0$ and let c_1 be equal to b_i, where i is the least j such that $b_j > b_0$. Then for $n \geq 1$, we inductively define:
(1) $c_{2n} = b_i$, where i is the least such that $b_i \notin \{c_0, c_1, \ldots, c_{2n-1}\}$, and
(2) $c_{2n+1} = b_k$, where k is the least such that $c_i < b_k$ for all $i \leq 2n$.

Now, consider the equivalence structure $\mathcal{A} = (\omega, E)$, where $E = \{(n,n) : n \in \omega\} \cup \{(c_{2n}, c_{2n+1}) : n \in \omega\}$. Then for each i, $card([c_i]) = 2$ and for

$a \notin B$, $[a] = \{a\}$. Thus, in $\mathcal{A}$, we have $B = \{a : card([a]) = 2\}$, as desired. It remains to show that E is a computable relation. Observe that $c_1 < c_3 < \cdots$, so for every n, $c_{2n+1} \geq n$. Now, given $a < b$, let $n = max\{a, b\}$. Then it easy to see that

$$aEb \iff (\exists m \leq n)[a = c_{2m} \wedge b = c_{2m+1}],$$

so E is computable. □

We note that it is easy to modify the proof of Theorem 4.1 to obtain an analogous result for computable equivalence structures which consist of infinitely many equivalence classes of size 1, infinitely many equivalence classes of size $k > 1$, and no other equivalence classes.

The analogue of Theorem 4.1 fails for structures with infinitely many classes of size k_1 and k_2 where $k_2 > k_1 > 1$. For example, we can prove the following.

Theorem 4.2. *There is a c.e. set B such that for any c.e. equivalence structure $\mathcal{A}$ with character $\{2, 3\} \times (\omega - \{0\})$, $\{a : card([a]) = 3\} \neq B$.*

Proof. Let $\mathcal{A}_e$ be the e-th equivalence structure. That is, $\mathcal{A}_e$ has universe ω and equivalence relation E_e which is the reflexive and transitive closure of the e-th c.e. set W_e. Thus, aE_eb holds if and only if

$$a = b \vee (\exists x_0, x_1, \ldots, x_k)[x_0 = a \;\&\; \ldots \;\&\; x_k = b \;\&\; (\forall i < k)(\langle x_i, x_{i+1}\rangle \in W_e)].$$

Let $[a]_e$ be the equivalence class of a in $\mathcal{A}_e$. Let $C_e = \{a : card([a]_e) = 3\}$. Let $E_{e,s}$ be the transitive closure of $W_{e,s}$ for all $e, s \geq 0$ and $\mathcal{A}_{e,s}$ be the equivalence structure $(\omega, E_{e,s})$.

We will construct a desired c.e. set B by a finite injury priority argument. Our construction will meet the following requirements for all $e \geq 0$.

Requirement R_e: If $\chi(\mathcal{A}_e) = \{2, 3\} \times (\omega - \{0\})$, then $C_e \neq B$.

To satisfy a particular requirement R_e, we find a pair a_e, b_e such that $a_eE_eb_e$, but $a_e \in B \iff b_e \notin B$.

At each stage s, we will define $a_{e,s}$ for each $e \geq 0$. We say the equivalence structure $\mathcal{A}_{e,s}$ is *active* at stage s as long as there are no equivalence classes of size > 3. If $\mathcal{A}_{e,s}$ is not active, then we will say that the requirement R_e in inactive, and that R_e is permanently satisfied for all stages $t \geq s$. For certain requirements $e < s$, we will also define $b_{e,s}$ such that $a_{e,s}E_{e,s}b_{e,s}$ and one of $a_{e,s}$, $b_{e,s}$ is in B_s, while the other is restrained by R_e from entering B. In this case, if A_e is active, then we say that the requirement R_e is *inactive*, and otherwise that R_e is *active*.

We say that requirement R_e with $e \leq s$ *requires attention* at stage $s+1$ if $\mathcal{A}_{e,s}$ and R_e are both active at stage s, $b_{e,s}$ is undefined, and there exists $b \leq s+1$ such that $b \neq a_{e,s}$ and $a_{e,s}E_{e,s}b$.

Construction

Stage 0: For each e, set $a_{e,0} = 2e$. Let $b_{e,0}$ be undefined for all $e \geq 0$.

Stage $s+1$: Let e be the least $f \leq s+1$ such that R_f requires attention at stage $s+1$. If there is no such e, then do nothing. Otherwise, let b be the least $z \leq s+1$ such that $z \neq a_{e,s}$ and $a_{e,s}E_{e,s}z$. Then we take actions following action.

Case I: Ler $b = a_{i,s}$ for some $i < e$. Then reset $a_{e,s+1}$ to be the least $x \neq b$ such that: (i) $x \notin B_s$, and (ii) $x \neq a_{j,s}$ for any j, and $x \neq b_{j,s}$ for any j for which $b_{j,s}$ is defined. Note that such x always exists since initially each $a_{k,s}$ is even, and only a finite number of odd elements will be used up to any stage s of the construction. We then let $a_{i,s+1} = a_{i,s}$ for all $i \neq e$, and let $b_{j,s} = b_{j,s+1}$ for all j such that $b_{j,s}$ is defined. Then go to stage $s+2$.

If we are not in Case I, then we know that $b \neq a_{i,s}$ for every $i \leq e$.

Case II: Let $b \notin B_s$. Then put $a_{e,s} \in B_{s+1}$, set $b_{e,s+1} = b$, and let R_e restrain b from entering B. We then let $a_{i,s+1} = a_{i,s}$ for all i, and let $b_{j,s} = b_{j,s+1}$ for all j such that $b_{j,s}$ is defined.

Case III: Let$b \in B_s$. In this case, we have R_e restrain $a_{e,s}$ from entering B and set $b_{e,s+1} = b$. We then let $a_{i,s+1} = a_{i,s}$ for all i, and let $b_{j,s} = b_{j,s+1}$ for all j such that $b_{j,s}$ is defined.

In either Case I or Case II, if $b = a_{i,s+1}$ for some $i > e$, then reset $a_{i,s+1}$ to be the least $x \neq b$ such that: (i) $x \notin B_s$, and (ii) $x \neq a_{j,s+1}$ for any $j \neq i$, and $x \neq b_{j,s+1}$ for any j such that $b_{j,s+1}$ is defined. Note that such x always exists since initially each $a_{k,s}$ is even, and only a finite number of odd elements will be used up to any stage s of the construction. Let $b_{i,s+1}$ become undefined. This might injure the requirement R_i.

This completes the construction.

Since each requirement R_e can only be injured by the (higher priority) requirements R_i with $i < e$, it is clear that R_e will require attention only a finite number of times. Thus, the limit $a_e = lim_s a_{e,s}$ exists for each e. Similarly, for each e, there exists a stage s_e such that for all $t \geq s_e$, either: (a) $b_{e,t}$ undefined at stage t, or (b) $b_{e,t} = b_{e,s_e}$, and $a_{e,t} = a_{e,s_e}$, and $a_{e,t} E_{e,t} b_{e,t}$, and $a_{e,t} \in B_{e,t} \iff b_e \notin B_{e,t}$. If $\mathcal{A}_e$ has characteristic $\{2,3\} \times (\omega - \{0\})$, then $card([a_e]) \geq 2$ and, hence, there exists b with $a_e E_e b$. Consider any stage s after which no action for any requirements R_i with $i \leq e$ will take place, and there is $b \leq s+1$ such that $a_e E_{e,s} b$. Then either $\mathcal{A}_e$ is inactive at stage $s+1$, in which case requirement R_e is permanently satisfied or $\mathcal{A}_e$ is active at stage $s+1$. In the second case, requirement R_e must be inactive at stage $s+1$ so that $b_{e,s+1}$ is defined, $a_{e,s+1} E_{e,s+1} b_{e,s+1}$, and $a_{e,s+1} \in B_{e,s+1} \iff b_{e,s+1} \notin B_{e,s+1}$. Since no requirement R_i with $i \leq e$ requires attention after stage $s+1$, we will never add either $a_{e,s+1}$ or $b_{e,s+1}$ to B after stage s, so that $a_{e,s+1}$ and $b_{e,s+1}$ will witness that $B \neq C_e$. □

Note that the c.e. set B constructed in this proof has the property that for any c.e. structure $\mathcal{A}$ with all equivalence classes of size ≥ 2 and for all $k \leq \omega$, $B \neq \{a : card([a]) = k\}$.

For equivalence structures with equivalence classes of three or more different cardinalities $k_1 < k_2 < \cdots < k_n$, the elements of an intermediate size equivalence class will form a d.c.e. set. Thus it is natural to ask whether any d.c.e. set can be represented in this way. Similar questions can be asked for Σ^0_1 and Π^0_1 equivalence structures.

5 Decidability of structures and theories

Recall that for any structure $\mathcal{A}$, $Th(\mathcal{A})$ denotes the first-order theory of $\mathcal{A}$, and $FTh(\mathcal{A})$ denotes the elementary diagram of $\mathcal{A}$. In this section, we consider the decidability of equivalence structures and their theories. The intuitive idea is that the character of an equivalence structure, together with the number of infinite classes, determines its theory. Similarly, the character, together with the function mapping any element to the size of its equivalence class, determines its elementary diagram.

Proposition 5.1. *If $Th(\mathcal{A})$ is decidable, then the character $\chi(\mathcal{A})$ is computable.*

Proof. It follows from the definition of $\chi(\mathcal{A})$ that the character is uniformly definable by first-order formulas. That is, it is easy to write down first-order formulas $\psi_{n,k}$ so that

$$(k, n) \in \chi(\mathcal{A}) \text{ if and only if } \mathcal{A} \models \psi_{n,k}.$$

□

It follows from the argument above that, in fact, $\chi(\mathcal{A})$ is many-one reducible to $Th(\mathcal{A})$. Define the set $K(\mathcal{A}) \subseteq \omega \times (\omega - \{0\})$ by

$$(a, k) \in K(\mathcal{A}) \iff card([a]) \geq k.$$

Theorem 5.2. *For any equivalence structure $\mathcal{A}$, the elementary diagram of $\mathcal{A}$ is Turing reducible to the join of the set $K(\mathcal{A})$ with the atomic diagram of $\mathcal{A}$.*

Proof. First, assume that $\mathcal{A}$ has only finitely many equivalence classes. Then, clearly, $FTh(\mathcal{A})$ is axiomatizable and hence computable. That is, for simplicity, let $a_1, \ldots, a_n$ be representatives of the n classes having cardinalities $k_1, \ldots, k_n$, respectively. Then in the expanded language with names for $a_1, \ldots, a_n$, we have the following axioms.

(i) Every element is equivalent to one of $a_1, \ldots, a_n$:

$$(\forall x)[xEa_1 \vee \ldots \vee xEa_n].$$

(ii) For every finite class $[a]$ with the representative a, there is an axiom giving the size k of the class:

$$(\exists x_1, \ldots, x_k)[\bigwedge_{i=1}^{k} x_i E a \ \wedge$$
$$\bigwedge_{\substack{i \neq j \\ 1 \leq i,j \leq k}} x_i \neq x_j \wedge (\forall z)(aEz \Longrightarrow (z = x_1 \vee \ldots \vee z = x_k))].$$

(iii) For every infinite class $[a]$, we have an axiom for every n:

$$(\forall x_1, \dots, x_n)(\exists y)[yEa \wedge y \neq x_1 \wedge \cdots \wedge y \neq x_n].$$

Now, we assume that $\mathcal{A}$ has infinitely many equivalence classes. We proceed by quantifier elimination. We first expand the language by adding the relation symbols γ_k such that $\mathcal{A} \models \gamma_k(a)$ if and only if $(a, k) \in K(\mathcal{A})$. Let $\psi(x, t_1, \dots, t_n)$ be any conjunction of literals in this expanded language, where $t_1, \dots, t_n$ are either variables or elements of $\mathcal{A}$, and let θ be $(\exists x)\psi$. Without loss of generality, we may assume that ψ includes either $t_i = t_j$ or $\neg(t_i = t_j)$ for all i, j where we set $x = t_0$. Similarly, we may assume that ψ includes either t_iEt_j or $\neg(t_iEt_j)$ for all i, j. As usual, it suffices to eliminate the quantifier from ψ. There are three cases.

Case 1. If ψ has a conjunct $x = t_i$ with $i > 0$, then θ is logically equivalent to the quantifier-free formula ψ^- obtained from ψ by replacing all occurrences of x with t_i.

In the remaining cases, ψ has the conjuncts $\neg(x = t_i)$ for all $i > 0$.

Case 2. Suppose that ψ has the conjunct xEt_m for some m. Let k be the number of distinct terms (modulo $\psi \models t_i = t_j$) out of $x, t_1, \dots, t_n$ such that t_iEt_m. Then θ is logically equivalent to the quantifier-free formula $\psi^- \wedge \gamma_{k+1}(t_m)$. That is, the desired x will exist if and only if $card([t_m]) \geq k+1$, so that $\mathcal{A}$ contains an additional element of $[t_m]$.

Case 3. Suppose that ψ has the conjuncts $\neg xEt_i$ for all i. Then, again, θ is equivalent to the formula ψ^-. This is true since $\mathcal{A}$ has infinitely many distinct equivalence classes.

At the end of quantifier elimination, we can determine whether the reduced formula ψ holds in $\mathcal{A}$ by consulting the diagram of $\mathcal{A}$ as well as $K(\mathcal{A})$. □

Theorem 5.3. *For any equivalence structure $\mathcal{B}$, there is a structure $\mathcal{A}$ isomorphic to $\mathcal{B}$, such that $\mathcal{A}$ and $K(\mathcal{A})$ are computable from $\chi(\mathcal{A})$.*

Proof. We may assume, without loss of generality, that $\mathcal{B}$ has no infinite equivalence classes, since, if needed, we can simply adjoin either infinitely many or some fixed finite number of infinite equivalence classes. We may also assume that $\mathcal{B}$ has infinitely many classes with at least two elements, since otherwise $\mathcal{B}$ certainly has a decidable copy. The structure $\mathcal{A}$ will contain a distinct equivalence class $[\langle k, n\rangle]$ for each $(k, n) \in \chi(\mathcal{B})$, where we let $\langle k, n\rangle = 2^{k+1} \cdot 3^{n+1}$. Let $\chi(\mathcal{B})$ be enumerated numerically as $\langle k_0, n_0\rangle, \langle k_1, n_1\rangle, \ldots$ and let $b_0, b_1, \ldots$ enumerate $\omega - \chi(\mathcal{B})$. Then $E = E^{\mathcal{A}}$ is defined by using the elements $b_0, b_1, \ldots$ to fill out the equivalence classes $[\langle k_0, n_0\rangle], [\langle k_1, n_1\rangle], \ldots$ in order, as needed. It is easy to see that $\mathcal{A}$ and $K(\mathcal{A})$ are computable from $\chi(\mathcal{A})$. □

Putting these results together, we have the next two theorems along with some immediate corollaries.

Theorem 5.4. *For any equivalence structure $\mathcal{A}$, $Th(\mathcal{A})$ and $\chi(\mathcal{A})$ have the same Turing degree.*

Proof. It follows from the argument in Proposition 5.1 that $\chi(\mathcal{A})$ is Turing reducible to $Th(\mathcal{A})$. Conversely, let $\mathcal{B}$ be an equivalence structure and let $\mathcal{A}$, isomorphic to $\mathcal{B}$, be given by Theorem 5.3, so that $\mathcal{A}$ and $K(\mathcal{A})$ are both computable from $\chi(\mathcal{A})$ (which, of course, equals $\chi(\mathcal{B})$). It follows from Theorem 5.2 that $FTh(\mathcal{A})$ is computable from $\chi(\mathcal{B})$. Now $Th(\mathcal{B}) = Th(\mathcal{A})$ is computable from $FTh(\mathcal{A})$, and, hence, is computable from $\chi(\mathcal{B})$ as desired. □

Corollary 5.5. *For any equivalence structure $\mathcal{A}$, $Th(\mathcal{A})$ is decidable if and only if $\chi(\mathcal{A})$ is computable.*

Theorem 5.6. *For any equivalence structure $\mathcal{B}$ with computable character $\chi(\mathcal{B})$, there is a decidable structure $\mathcal{A}$ isomorphic to $\mathcal{B}$. (Hence $Th(\mathcal{B})$ is decidable.)*

Proof. Again, it suffices to assume that $\mathcal{B}$ has no infinite equivalence classes. By Theorem 5.3, there is a structure $\mathcal{A}$ isomorphic to $\mathcal{B}$, which is computable from $\chi(\mathcal{A})$, and hence $\mathcal{A}$ and $K(\mathcal{A})$ are also computable. It now follows from Theorem 5.2 that $FTh(\mathcal{A})$ is decidable, and hence $Th(\mathcal{A})$, which equals $Th(\mathcal{B})$, is decidable. □

Clearly, any bounded character is computable.

Corollary 5.7. *If the equivalence structure $\mathcal{A}$ has bounded character, then $Th(\mathcal{A})$ is decidable.*

For computably categorical structures, we can say more.

Corollary 5.8. *If $\mathcal{A}$ is a computably categorical equivalence structure, then $\mathcal{A}$ is decidable.*

Proof. Let $\mathcal{A}$ be computably categorical. Then $\mathcal{A}$ has bounded character, so $\chi(A)$ is computable. Hence by Theorem 5.6, there is a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, which is decidable. Since $\mathcal{A}$ is computably categorical, $\mathcal{A}$ is computably isomorphic to $\mathcal{B}$ and, therefore, $\mathcal{A}$ is also decidable. □

Note that there are equivalence structures that are not computably categorical, which have decidable theories. For example, fix $k_1 < k_2 \leq \omega$ and let $\mathcal{A}$ have infinitely many equivalence classes of size k_1 and infinitely many classes of size k_2 and no other classes. Then $\chi(\mathcal{A})$ is computable and, thus, $Th(\mathcal{A})$ is decidable. We note that in all considered cases of decidable theories, one could, in fact, give a complete set of axioms for the theory.

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