ALGORITHMIC RANDOMNESS AND CAPACITY OF CLOSED SETS

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Abstract. We investigate the connection between measure, capacity and algorithmic randomness for the space of closed sets. For any computable measure m, a computable capacity T may be defined by letting $T(Q)$ be the measure of the family of closed sets K which have nonempty intersection with Q. We prove an effective version of Choquet's capacity theorem by showing that every computable capacity may be obtained from a computable measure in this way. We establish conditions on the measure m that characterize when the capacity of an m-random closed set equals zero. This includes new results in classical probability theory as well as results for algorithmic randomness. For certain computable measures, we construct effectively closed sets with positive capacity and with Lebesgue measure zero. We show that for computable measures, a real q is upper semi-computable if and only if there is an effectively closed set with capacity q.

Introduction

The study of algorithmic randomness has been an active area of research in recent years. The basic problem is to quantify the randomness of a single real number. Here we think of a real $r \in [0,1]$ as an infinite sequence of 0's and 1's, i.e. as an element in $2^{\mathbb{N}}$. There are three basic approaches to algorithmic randomness: the measure-theoretic approach of Martin-Löf tests, the incompressibility approach of Kolmogorov complexity, and the betting approach in terms of martingales. All three approaches have been shown to yield the same notion of (algorithmic) randomness. The present paper will consider only the measure-theoretic approach. A real x is Martin-Löf random if for any effective sequence $S_1, S_2, \dots$ of c. e. open

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sets with $\mu(S_n) \leq 2^{-n}$, $x \notin \bigcap_n S_n$. For background and history of algorithmic randomness we refer to [8, 15].

The study of random sets and in particular of random closed sets is a vibrant area in probability and statistics, with many applications in science and engineering. The notion of capacity plays an important role here as a part of the analysis of imprecise or uncertain observations, for example in intelligent systems. For background on the theory of random sets see [14].

In a series of recent papers [4, 2], G. Barmpalias, P. Brodhead, D. Cenzer, S. Dashti, J.B. Remmel and R. Weber have defined a notion of algorithmic randomness for closed sets and continuous functions on $2^{\mathbb{N}}$. Here the Polish space $2^{\mathbb{N}}$ is equipped with usual product topology and has a basis of clopen sets. Definitions are given below in section 1. The space $\mathcal{C}$ of closed subsets of $2^{\mathbb{N}}$ has the *hit-or-miss* or *Fell* topology which is also described in section 1. In general when we discuss closed sets in this paper we are refering to closed subsets of $2^{\mathbb{N}}$.

The study of randomness for closed sets and continuous functions has several interesting aspects concerning properties of those sets and properties of the members of such sets. The topological and measure-theoretic properties of effectively random closed sets has been studied. For example, it is shown in [4] that every effectively random closed set is perfect and has Lebesgue measure 0. The complexity of effectively random closed sets as subsets of $2^{\mathbb{N}}$ was considered in [4], where it was shown that no effectively closed (Π^0_1) set is random but there is a random Δ^0_2 closed set.

The members of a closed set are reals and hence we can study the complexity of the members of an effectively random closed set. The following results were obtained in [4]. Every effectively random closed set contains a random member but not every member is random. Every random real belongs to some random closed set. Every effectively random Δ^0_2 closed set contains a random Δ^0_2 member. Effectively random closed set contain no computable elements (in fact, no n-c. e. elements). It was shown in [2] that the set of zeroes of an effectively random continuous function is an effectively random closed set.

Just as an effectively closed set in $2^{\mathbb{N}}$ may be viewed as the set of infinite paths through a computable tree $T \subseteq \{0,1\}^*$, an algorithmically random closed set in $2^{\mathbb{N}}$ may be viewed as the set of infinite paths through an algorithmically random tree T. Diamondstone and Kjos-Hanssen [11, 10] give an alternative definition of algorithmic randomness for closed sets according to the Galton-Watson distribution and show that this definition produces the same family of algorithmically random closed sets.

We note that in probability theory a random closed subset of a topological space X is considered a random variable which takes on the values in the space $\mathcal{C}(X)$ of closed subsets of X. That is, let $(\Omega, \mathcal{A}, P)$ be a *probability space* with underlying topological space Ω, σ-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ and measure P such that $P(\Omega) = 1$ and $P(S)$ is defined for all sets $S \in \mathcal{A}$. For example, we might have $\Omega = 2^{\mathbb{N}}$, $\mathcal{A}$ the family of Borel subsets of $2^{\mathbb{N}}$, and P the standard Lebesgue measure. The map X induces a probability measure P_X on $\mathcal{C}(X)$ given by $P(X^{-1}(S)$. Classically, the statement that a random closed set has no computable elements means that the collection of closed sets with no computable elements has measure one. In effective randomness, there is a particular collection R of *algorithmically random* closed sets which has measure one. In this context, the statement that effectively random closed sets have no computable elements is to say that the closed sets in R have no computable elements. The latter result of course implies the former, but is stronger.

A random closed set is a specific type of random recursive construction, as studied by Graf, Mauldin and Williams [9]. McLinden and Mauldin [13] showed that the Hausdorff dimension of a random closed set is $log_2(4/3)$, that is, almost every closed subset of $2^{\mathbb{N}}$ has Hausdorff dimension $log_2(4/3)$. It was shown in [4] that every effectively random closed set has box dimension $log_2(4/3)$. The *effective* Hausdorff dimension of members of effectively random closed sets is studied in [11]. It is shown that every member of an effectively random closed set has effective Hausdorff dimension $\geq log_2(3/2)$ and that any real with effective Hausdorff dimension $> log_2(3/2)$ is a member of some effectively random closed set.

In the present paper we will examine the notion of computable capacity and its relation to computable measures on the space $\mathcal{C}$ of nonempty closed sets. Given a domain U, a *capacity* $\mathcal{T}$ is a real-valued function defined on some σ-field of subsets of U, which is closely related to measure. $\mathcal{T}$ may be thought of as a *belief* function in the context of reasoning with uncertainty. (See [14, p. 71] and also [18].) The capacity $\mathcal{T}(A)$ for a set A is the probability that a randomly chosen set S is a subset of A.

Choquet [6] developed the Choquet capacity for the space $\mathcal{C}$ of closed subsets of an infinite set X. A probability measure μ^* on $\mathcal{C}$ induces a capacity $\mathcal{T}$ on $\mathcal{C}$ by defining the capacity $\mathcal{T}(C)$ of a closed set C to be $\mu^*(\{K \in \mathcal{C} : K \subseteq C\})$. Choquet's capacity theorem states that every capacity $\mathcal{T}$ on $\mathcal{C}$ arises in this way from some measure μ^*.

In section one, we give some basic definitions including the definition of the space of $\mathcal{C}(X)$ of closed subsets of a computable Polish space X. We present a family of computable measures on $\mathcal{C}$ which will lead to different notions of effective randomness for closed sets.

In section two, we define the notion of computable capacity and show how a measure on the space of closed sets induces a capacity. An effective version of Choquet's capacity theorem is proved.

The main theorem of section three gives conditions under which the capacity $\mathcal{T}(Q)$ of a μ^*-random closed set Q is either equal to 0 or > 0. In particular, suppose that the measure μ_b on $\{0,1,2\}^{\mathbb{N}}$ is defined so that, for all $\sigma \in \{0,1,2\}^*$, $\mu_b(I(\sigma^\frown i)) = b \cdot \mu_b(I(\sigma))$ for $i = 0, 1$ and define the corresponding probability measure μ_b^* and capacity $\mathcal{T}_b$ on the space $\mathcal{C}$ of closed sets and the corresponding capacity $\mathcal{T}_b$. This means that for any node σ in the tree T_Q, σ has unique extension $\sigma^\frown 0$ in T_Q with probability b, and similarly σ has unique extension $\sigma^\frown 1$ with probability b. Then we show the following. If $b \geq 1 - \frac{\sqrt{2}}{2}$, then every effectively μ_b^*-random closed set Q has capacity $\mathcal{T}_b(Q) = 0$. It is important to note that, since the random closed sets have measure one in the space $\mathcal{C}$ of closed sets, this result implies that almost all closed sets have capacity zero. This is a *new* result about the classical measure and capacity of closed sets in general and not only about algorithmic randomness or computability.

On the other hand, if $b < 1 - \frac{\sqrt{2}}{2}$, then every effectively μ_b^*-random closed set Q has capacity $\mathcal{T}_b(Q) > 0$, and hence almost every closed set has positive capacity. A more general result is given.

In section four, we consider the capacity of effectively closed sets. Fix computable reals b_0 and b_1 such that $0 < b_1 \leq b_0$ and $b_0 + b_1 < 1$ and define the measure μ on $\{0,1,2\}^{\mathbb{N}}$ so that for any $\sigma \in \{0,1,2\}^*$ and for $i \in \{0,1\}$, $\mu(I(\sigma^\frown i)) = b_i \cdot \mu(I(\sigma))$. Let μ^* be the corresponding measure on $\mathcal{C}$ and let $\mathcal{T}$ be the corresponding capacity. It is easy to see that for any effectively closed set Q, $\mathcal{T}(Q)$ is an upper-semi-computable real. Conversely, for any upper-semi-computable real q, there exists an effectively closed set Q with capacity $T(Q) = q$. We also show that if $b_0 = b_1$, there exists an effectively closed set Q with Lebesgue measure zero and with positive capacity.

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1. Computable Measures on the Space of Closed Sets

We present an effective version of Choquet's theorem connecting measure and capacity.

In this section, we describe the hit-or-miss topology on the space $\mathcal{C}$ of closed sets, we define certain probability measures μ_d on the space $\{0,1,2\}^{\mathbb{N}}$ and the corresponding measures μ_d^* on the homeomorphic space $\mathcal{C}$. These give rise to notions of algorithmic randomness for closed sets.

Some definitions are needed. For a finite string $\sigma \in \{0,1\}^n$, let $|\sigma| = n$. Let λ denote the empty string so that $|\lambda| = 0$. For two strings σ, τ, say that σ is an *initial segment* of τ and write $\sigma \sqsubseteq \tau$ if $|\sigma| \leq |\tau|$ and $\sigma(i) = \tau(i)$ for $i < |\sigma|$. For $x \in 2^{\mathbb{N}}$, $\sigma \sqsubset x$ means that $\sigma(i) = x(i)$ for $i < |\sigma|$. Let $\sigma^\frown\tau$ denote the concatenation of σ and τ and let $\sigma^\frown i$ denote $\sigma^\frown(i)$ for $i = 0, 1$. For $\sigma \in \{0,1\}^*$ and $x \in 2^{\mathbb{N}}$, $\sigma^\frown x = (\sigma(0), \dots, \sigma(|\sigma|-1), x(0), x(1), \dots)$. Let $x\lceil n = (x(0), \dots, x(n-1))$. Two reals x and y may be coded together into $z = x \oplus y$, where $z(2n) = x(n)$ and $z(2n+1) = y(n)$ for all n. For a finite string σ, let $I(\sigma)$ denote $\{x \in 2^{\mathbb{N}} : \sigma \sqsubset x\}$. We shall call $I(\sigma)$ the *interval* determined by σ. Each such interval is a clopen set and the clopen sets are just finite unions of intervals. We let $\mathcal{B}$ denote the computable Boolean algebra of clopen sets. Note that this is a countable atomless Boolean algebra.

A set $T \subseteq \{0,1\}^*$ is a *tree* if it is closed under initial segments. For an arbitrary tree $T \subseteq \{0,1\}^*$, let $[T]$ denote the set of infinite paths through T. It is well-known that $P \subseteq 2^{\mathbb{N}}$ is a closed set if and only if $P = [T]$ for some tree T. P is a Π^0_1 class, or an effectively closed set, if $P = [T]$ for some computable tree T.

A closed set P may be identified with a tree $T_P \subseteq \{0,1\}^*$ where $T_P = \{\sigma : P \cap I(\sigma) \neq \emptyset\}$. Note that T_P has no dead ends. That is, if $\sigma \in T_P$, then either $\sigma^\frown 0 \in T_P$ or $\sigma^\frown 1 \in T_P$. The complexity of the closed set P is generally identified with that of T_P. Thus P is said to be a Π^0_2 closed set if T_P is Π^0_2; in this case $P = [T]$ for some Δ^0_2 tree T. The complement of an effectively closed set is sometimes called a c. e. open set. We remark that if P is an effectively closed set, then T_P is a Π^0_1 set, but it is not, in general, computable. For any $\sigma \in \{0,1\}^*$ and any $Q \subseteq 2^{\mathbb{N}}$, we let $\sigma^\frown Q$ denote $\{\sigma^\frown x : x \in Q\}$. There is a natural effective enumeration $P_0, P_1, \dots$ of the effectively closed sets and thus an enumeration of the c. e. open sets. Thus we can say that a sequence $S_0, S_1, \dots$ of c. e. open sets is *effective* if there is a computable function, f, such that $S_n = 2^{\mathbb{N}} - P_{f(n)}$ for all n. For a detailed development of effectively closed sets, see [5].

It was observed in [4] that there is a natural isomorphism between the space $\mathcal{C}$ of nonempty closed subsets of $\{0,1\}^{\mathbb{N}}$ and the space $\{0,1,2\}^{\mathbb{N}}$ (with the product topology) defined as follows. Given a nonempty closed $Q \subseteq 2^{\mathbb{N}}$, let $T = T_Q$ be the tree without dead ends such that $Q = [T]$. Let $\sigma_0, \sigma_1, \dots$ enumerate the elements of T in order, first by length and then lexicographically. We then define the code $x = x_Q = x_T$ by recursion such that for each n, $x(n) = 2$ if both $\sigma_n{}^\frown 0$ and $\sigma_n{}^\frown 1$ are in T, $x(n) = 1$ if $\sigma_n{}^\frown 0 \notin T$ and $\sigma_n{}^\frown 1 \in T$, and $x(n) = 0$ if $\sigma_n{}^\frown 0 \in T$ and $\sigma_n{}^\frown 1 \notin T$. For a finite tree $T \subseteq \{0,1\}^{\leq n}$, the finite code ρ_T is similarly defined, ending with $\rho_T(k)$ where σ_k is the lexicographically last element of $T \cap \{0,1\}^{\leq n}$.

We defined in [4] a measure μ^* on the space $\mathcal{C}$ of closed subsets of $2^{\mathbb{N}}$ as follows.

$$\mu^*(\mathcal{X}) = \mu(\{x_Q : Q \in \mathcal{X}\}) \tag{1.1}$$

for any $\mathcal{X} \subseteq \mathcal{C}$ and μ is the standard measure on $\{0,1,2\}^{\mathbb{N}}$. Informally this means that given $\sigma \in T_Q$, there is probability $\frac{1}{3}$ that both $\sigma^\frown 0 \in T_Q$ and $\sigma^\frown 1 \in T_Q$ and, for $i = 0, 1$, there is probability $\frac{1}{3}$ that only $\sigma^\frown i \in T_Q$. In particular, this means that $Q \cap I(\sigma) \neq \emptyset$ implies that for $i = 0, 1$, $Q \cap I(\sigma^\frown i) \neq \emptyset$ with probability $\frac{2}{3}$.

Then we say that a closed set $Q \subseteq 2^{\mathbb{N}}$ is (Martin-Löf) random if x_Q is (Martin-Löf) random. Note that the equal probability of $\frac{1}{3}$ for the three cases of branching allows the application of Schnorr's theorem that Martin-Löf randomness is equivalent to prefix-free Kolmogorov randomness.

The standard (*hit-or-miss*) topology [7, p. 45] on the space $\mathcal{C}$ of closed sets is given by a sub-basis of sets of two types, where U is any open set in $2^{\mathbb{N}}$.

$$V(U) = \{K : K \cap U \neq \emptyset\}; \qquad W(U) = \{K : K \subseteq U\}$$

Note that $W(\emptyset) = \{\emptyset\}$ and that $V(2^{\mathbb{N}}) = \mathcal{C} \setminus \{\emptyset\}$, so that $\emptyset$ is an isolated element of $\mathcal{C}$ under this topology. Thus we may omit $\emptyset$ from $\mathcal{C}$ without complications.

A basis for the hit-or-miss topology may be formed by taking finite intersections of the basic open sets. We want to work with the following simpler basis. For each n and each finite tree $A \subseteq \{0,1\}^{\leq n}$, let

$$U_A = \{K \in \mathcal{C} : (\forall \sigma \in \{0,1\}^{\leq n})\ (\sigma \in A \iff K \cap I(\sigma) \neq \emptyset)\}.$$

That is,

$$U_A = \{K \in \mathcal{C} : T_K \cap \{0,1\}^{\leq n} = A\}.$$

Note that the sets U_A are in fact clopen. That is, for any tree $A \subseteq \{0,1\}^{\leq n}$, define the tree $A' = \{\sigma \in \{0,1\}^{\leq n} : (\exists \tau \in \{0,1\}^n \setminus A) \sigma \sqsubseteq \tau\}$. Then $U_{A'}$ is the complement of U_A.

For any finite n and any tree $T \subseteq \{0,1\}^{\leq n}$, define the clopen set $[T] = \bigcup_{\sigma \in T} I(\sigma)$. Then $K \cap [T] \neq \emptyset$ if and only if there exists some $A \subseteq \{0,1\}^{\leq n}$ such that $K \in U_A$ and $A \cap T \neq \emptyset$. That is,

$$V([T]) = \bigcup \{U_A : A \cap T \neq \emptyset\}.$$

Similarly, $K \subseteq [T]$ if and only if there exists some $A \subseteq \{0,1\}^n$ such that $K \in U_A$ and $A \subseteq T$. That is,

$$W([T]) = \bigcup \{U_A : A \subseteq T\}.$$

The following lemma can now be easily verified.

Lemma 1. *The family of sets* $\{U_A : A \subseteq \{0,1\}^{\leq n}\ A \text{ is a tree}\}$ *is a basis of clopen sets for the hit-or-miss topology on* $\mathcal{C}$.

Recall the mapping from $\mathcal{C}$ to $\{0,1,2\}^{\mathbb{N}}$ taking Q to x_Q. It can be shown that this is in fact a homeomorphism. (See Axon [1] for details.) Let $\mathcal{B}^*$ be the family of clopen subsets of $\mathcal{C}$; each set is a finite union of basic sets of the form U_A and thus $\mathcal{B}^*$ is a computable atomless Boolean algebra. Note that elements U of $\mathcal{B}^*$ are collections of closed sets and are closed and open in the hit-or-miss topology on the space $\mathcal{C}$ of closed subsets of $\{0,1\}^{\mathbb{N}}$. Recall that $\mathcal{B}$ denotes the family of clopen subsets of $\{0,1\}^{\mathbb{N}}$.

Proposition 2. *The space* $\mathcal{C}$ *of nonempty closed subsets of* $2^{\mathbb{N}}$ *is computably homeomorphic to the space* $\{0,1,2\}^{\mathbb{N}}$. *Furthermore, the corresponding map from* $\mathcal{B}$ *to* $\mathcal{B}^*$ *is a computable isomorphism of these computable Boolean algebras.*

Next we consider probability measures μ on the space $\{0,1,2\}^{\mathbb{N}}$ and the corresponding measures μ^* on $\mathcal{C}$ induced by μ.

A probability measure on $\{0,1,2\}^{\mathbb{N}}$ may be defined as in [16] from a function $d : \{0,1,2\}^* \to [0,1]$ such that $d(\lambda) = 1$ and, for any $\sigma \in \{0,1,2\}^*$,

$$d(\sigma) = \sum_{i=0}^{2} d(\sigma^\frown i).$$

The corresponding measure μ_d on $\{0,1,2\}^{\mathbb{N}}$ is then defined by letting $\mu_d(I(\sigma)) = d(\sigma)$. Since the intervals $I(\sigma)$ form a basis for the standard product topology on $\{0,1,2\}^{\mathbb{N}}$, this will extend to a measure on all Borel sets. If d is computable, then μ_d is said to be computable. The measure μ_d is said to be *nonatomic* or *continuous* if $\mu_d(\{x\}) = 0$ for all $x \in \{0,1,2\}^{\mathbb{N}}$. We will say that μ_d is *bounded* if there exist bounds $b, c \in (0,1)$ such that, for any $\sigma \in \{0,1,2\}^*$ and $i \in \{0,1,2\}$,

$$b \cdot d(\sigma) < d(\sigma^\frown i) < c \cdot d(\sigma).$$

It is easy to see that any bounded measure must be continuous. We will say that the measure μ_d is *uniform* if there exist constants b_0, b_1, b_2 with $b_0 + b_1 + b_2 = 1$ such that for all σ and for $i \leq 2$, $d(\sigma^\frown i) = b_i \cdot d(\sigma)$.

Now let μ_d^* be defined by

$$\mu_d^*(\mathcal{X}) = \mu_d(\{x_Q : Q \in \mathcal{X}\}).$$

Let us say that a measure μ^* on $\mathcal{C}$ is computable if the restriction of μ^* to the family $\mathcal{B}^*$ of clopen sets is computable. That is, if there is a computable function F mapping $\mathcal{B}^*$ to $[0,1]$ such that $F(B) = \mu^*(B)$ for all $B \in B^*$.

Proposition 3. *For any computable d, the measure μ_d^* is a computable measure on $\mathcal{C}$.*

Proof. For any tree $A \subseteq \{0,1\}^{\leq n}$, it is easy to see that

$$K \in U_A \iff \rho_A \sqsubset x_K,$$

so that $\mu_d^*(U_A) = \mu_d(I(\rho_A))$. □

2. Computable Capacity and Choquet's Theorem

In this section, we define the notion of capacity and of computable capacity. We present an effective version of Choquet's theorem connecting measure and capacity. For details on capacity and random set variables, see Nguyen [14] and also Matheron [12].

Definition 4. *A* capacity *on $\mathcal{C}$ is a function $\mathcal{T} : \mathcal{C} \to [0,1]$ with $\mathcal{T}(\emptyset) = 0$ such that*

(1) $\mathcal{T}$ is monotone increasing, that is,

$$Q_1 \subseteq Q_2 \longrightarrow \mathcal{T}(Q_1) \leq \mathcal{T}(Q_2).$$

(2) $\mathcal{T}$ has the alternating of infinite order *property, that is, for $n \geq 2$ and any $Q_1, \dots, Q_n \in \mathcal{C}$*

$$\mathcal{T}(\bigcap_{i=1}^{n} Q_i) \leq \sum \{(-1)^{|I|+1} \mathcal{T}(\bigcup_{i \in I} Q_i) : \emptyset \neq I \subseteq \{1,2,\dots,n\}\}.$$

(3) If $Q = \bigcap_n Q_n$ and $Q_{n+1} \subseteq Q_n$ for all n, then $\mathcal{T}(Q) = \lim_{n\to\infty} \mathcal{T}(Q_n)$.

We will also assume, unless otherwise specified, that the capacity $\mathcal{T}(2^{\mathbb{N}}) = 1$.

We will say that a capacity $\mathcal{T}$ is computable if it is computable on the family of clopen sets, that is, if there is a computable function F from the Boolean algebra $\mathcal{B}$ of clopen sets into $[0,1]$ such that $F(B) = \mathcal{T}(B)$ for any $B \in \mathcal{B}$.

Define $\mathcal{T}_d(Q) = \mu_d^*(V(Q))$. That is, $\mathcal{T}_d(Q)$ is the probability that a randomly chosen closed set meets Q. Here is the first result connecting effective measure and effective capacity. This follows easily from the classical proof of Choquet.

Theorem 5. *If μ_d^* is a (computable) probability measure on $\mathcal{C}$, then $\mathcal{T}_d$ is a (computable) capacity.*

Proof. Certainly $\mathcal{T}_d(\emptyset) = 0$. The alternating property follows by basic probability. For (iii), suppose that $Q = \bigcap_n Q_n$ is a decreasing intersection. Then by compactness, $Q \cap K \neq \emptyset$ if and only if $Q_n \cap K \neq \emptyset$ for all n. Furthermore, $V(Q_{n+1}) \subseteq V(Q_n)$ for all n. Thus

$$\mathcal{T}_d(Q) = \mu_d^*(V(Q)) = \mu_d^*(\bigcap_n V(Q_n)) = \lim_n \mu_d^*(V(Q_n)) = \lim_n \mathcal{T}_d(Q_n).$$

If d is computable, then $\mathcal{T}_d$ may be computed as follows. For any clopen set $I(\sigma_1) \cup \cdots \bigcup I(\sigma_k)$ where each $\sigma_i \in \{0,1\}^n$, we compute the probability distribution for all trees of height n and add the probabilities of those trees which contain one of the σ_i. □

Choquet's Capacity Theorem states that any capacity $\mathcal{T}$ is determined by a measure, that is $\mathcal{T} = \mathcal{T}_d$ for some d. See [14] for details. We now give an effective version of Choquet's theorem. It is not so easy, but this does follow from the classical proof of Choquet [6]. See also [12] and Axon [1].

Theorem 6 (Effective Choquet Capacity Theorem). *If $\mathcal{T}$ is a computable capacity, then there is a computable measure μ_d^* on the space of closed sets such that $\mathcal{T} = \mathcal{T}_d$.*

Proof. Given the values $\mathcal{T}(U)$ for all clopen sets $I(\sigma_1) \cup \cdots \bigcup I(\sigma_k)$ where each $\sigma_i \in \{0,1\}^n$, there is in fact a unique probability measure μ_d on these clopen sets such that $\mathcal{T} = \mathcal{T}_d$ and this can be computed as follows.

Suppose first that $\mathcal{T}(I(i)) = a_i$ for $i < 2$ and note that each $a_i \leq 1$ and $a_0 + a_1 \geq 1$ by the alternating property. If $\mathcal{T} = \mathcal{T}_d$, then we must have $d((0)) + d((2)) = a_0$ and $d((1)) + d((2)) = a_1$ and also $d((0)) + d((1)) + d((2)) = 1$, so that $d((2)) = a_0 + a_1 - 1$, $d((0)) = 1 - a_1$ and $d((1)) = 1 - a_0$. This will imply that $\mathcal{T}(I\tau))) = \mathcal{T}_d(I(\tau))$ when $|\tau| = 1$. Now suppose that we have defined $d(\tau)$ and that τ is the code for a finite tree with elements $\sigma_0, \ldots, \sigma_n = \sigma$ and thus $d(\tau^\frown i)$ is giving the probability that σ will have one or both immediate successors. We proceed as above. Let $\mathcal{T}(I(\sigma^\frown i)) = a_i \cdot \mathcal{T}(I(\sigma))$ for $i < 2$. Then as above $d(\tau^\frown 2) = d(\tau) \cdot (a_0 + a_1 - 1)$ and $d(\tau^\frown i) = d(\tau) \cdot (1 - a_i)$ for each i. □

3. Zero Capacity

In this section, we compute the capacity of a random closed set under certain computable probability measures. In particular, suppose that μ_d is a symmetric measure, that is, let $d(\sigma^\frown 0) = d(\sigma^\frown 1)$ for all σ. We show the following. If $d(\sigma^\frown 2) \leq \frac{\sqrt{2}}{2} d(\sigma)$ for all σ, then $\mathcal{T}_d(R) = 0$ for any μ_d^*-random closed set R. Thus for the uniform measure with $d(\sigma^\frown 0) = d(\sigma^\frown 1) = \frac{1}{3} \cdot d(\sigma)$ for all σ, effectively random closed sets have capacity zero. Thus for almost all closed sets R, $\mathcal{T}_d(R) = 0$. If $d(\sigma^\frown 2) \geq b \cdot d(\sigma)$ for all σ, where $b > \frac{\sqrt{2}}{2}$

is a constant, then $\mathcal{T}_d(R) > 0$ for any μ_d^*-random closed set R. Thus for almost all closed sets R, $\mathcal{T}_d(R) > 0$. This result, and others in this section are new results about classical measure and capacity as well as results about algorithmic randomness.

For non-symmetric measures, where $d(\sigma^\frown i) = b_i \cdot d(\sigma)$ for $i < 2$, the question of whether a random closed set has zero capacity depends on the sum b_0+b_1 and also on their difference. If $b_0 + b_1 \geq 2 - \sqrt{2}$ and $|b_0 - b_1|$ is sufficiently small, then every μ_d^*-random closed set will have capacity zero (so that for almost all closed sets R, $\mathcal{T}_d(R) = 0$) and otherwise there is a μ_d^*-random closed set with positive capacity.

We say that $K \in \mathcal{C}$ is *μ_d^*-random* if x_K is Martin-Löf random with respect to the measure μ_d. (See [16] for details.) Our results show that the $\mathcal{T}_d$ capacity of a μ_d^*-random closed set depends on the particular measure.

In the following proofs, the key idea is that an arbitrary closed set Q can be given as the intersection of a sequence $\langle Q_n \rangle_{n\in\omega}$ of a sequence of clopen sets, so that the capacity $\mathcal{T}(Q) = \lim_n \mathcal{T}(Q_n)$. Thus we want to compute the capacity q_n of Q_n when Q is a random closed set, or at least to compute bounds on this capacity. Now the capacity of Q is the probability that $Q \cap K \neq \emptyset$ for a random closed set K, that is to say $\mathcal{T}(Q) = \mu_d^*(\{K : Q \cap K \neq \emptyset\})$. Thus we first compute the probability that $Q_n \cap K_n \neq \emptyset$ for randomly chosen closed sets Q and K and use this to determine $\mathcal{T}(Q)$ for a random closed set. In the first two theorems, these computations can be converted into Martin-Löf tests, so that the capacity of an effectively μ_d^*-random closed set can be determined.

Theorem 7. *Suppose that the measure μ_d is defined by d such that, for all sufficiently long $\sigma \in \{0,1\}^*$, $d(\sigma^\frown 2) \leq \frac{\sqrt{2}}{2} d(\sigma)$ and $d(\sigma^\frown 0) = d(\sigma^\frown 1)$. Then, for any μ_d^*-random closed set R, $\mathcal{T}_d(R) = 0$. Thus for almost all closed sets R, $\mathcal{T}_d(R) = 0$.*

Proof. We first present the proof for a uniform measure μ_d and then give the modifications necessary for non-uniform measure.

Fix b with $1 - 2b \leq \frac{\sqrt{2}}{2}$ and suppose that, for all σ, $d(\sigma^\frown 2) = (1-2b) \cdot d(\sigma)$ and, for $i = 0, 1$, $d(\sigma^\frown i) = b \cdot d(\sigma)$. Now let $\mu^* = \mu_d^*$. We will compute the probability, given two closed sets Q and K, that $Q \cap K$ is nonempty. Here we define the usual product measure on the product space $\mathcal{C} \times \mathcal{C}$ of pairs (Q, K) of nonempty closed sets by letting $\mu^2(U_A \times U_B) = \mu^*(U_A) \cdot \mu^*(U_B)$ for arbitrary subsets A, B of $\{0,1\}^n$.

Let

$$Q_n = \bigcup \{I(\sigma) : \sigma \in \{0,1\}^n \ \& \ Q \cap I(\sigma) \neq \emptyset\}$$

and similarly for K_n. Then $Q \cap K \neq \emptyset$ if and only if $Q_n \cap K_n \neq \emptyset$ for all n. Let p_n be the probability that $Q_n \cap K_n \neq \emptyset$ for two arbitrary closed sets K and Q, relative to our measure μ^*. It is immediate that $p_1 = 1 - 2b^2$, since $Q_1 \cap K_1 = \emptyset$ only when $Q_1 = I(i)$ and $K_1 = I(1-i)$. Next we will determine the quadratic function f such that $p_{n+1} = f(p_n)$. There are 9 possible cases for Q_1 and K_1, which break down into 4 distinct cases in the computation of p_{n+1}.

Case (i): As we have seen, $Q_1 \cap K_1 = \emptyset$ with probability $1 - 2b^2$.

Case (ii): There are two chances that $Q_1 = K_1 = I(i)$, each with probability b^2 so that $Q_{n+1} \cap K_{n+1} \neq \emptyset$ with probability p_n.

Case (iii): There are four chances where $Q_1 = 2^{\mathbb{N}}$ and $K_1 = I(i)$ or vice versa, each with probability $b \cdot (1-2b)$, so that once again $Q_{n+1} \cap K_{n+1} \neq \emptyset$ with relative probability p_n.

Case (iv): There is one chance that $Q_1 = K_1 = 2^{\mathbb{N}}$, with probability $(1-2b)^2$, in which case $Q_{n+1} \cap K_{n+1} \neq \emptyset$ with relative probability $1-(1-p_n)^2 = 2p_n - p_n^2$. This is because $Q_{n+1} \cap K_{n+1} = \emptyset$ if and only if both $Q_{n+1} \cap I(i) \cap K_{n+1} = \emptyset$ for both $i = 0$ and $i = 1$.

Adding these cases together, we see that

$$p_{n+1} = [2b^2 + 4b(1-2b)]p_n + (1-2b)^2(2p_n - p_n^2) = (2b^2 - 4b + 2)p_n - (1 - 4b + 4b^2)p_n^2.$$

Next we investigate the limit of the computable sequence $\langle p_n \rangle_{n \in \omega}$. Let $f(p) = (2b^2 - 4b+2)p - (1-4b+4b^2)p^2$. Note that $f(0) = 0$ and $f(1) = 1 - 2b^2 < 1$. It is easy to see that the fixed points of f are $p = 0$ and $p = \frac{2b^2-4b+1}{(1-2b)^2}$. Note that since $b < \frac{1}{2}$, the denominator is not zero and hence is always positive.

Now consider the function $g(b) = 2b^2 - 4b + 1 = 2(b-1)^2 - 1$, which has positive root $\hat{b} = 1 - \frac{\sqrt{2}}{2}$ and is decreasing for $0 \leq b \leq 1$.

There are three cases to consider when comparing b with $\hat{b}$.

Case 1: If $b > \hat{b}$, then $g(b) < 0$ and hence the other fixed point of f is negative. Furthermore, $2b^2 - 4b + 2 < 1$ so that $f(p) < p$ for all $p > 0$. It follows that the sequence $\{p_n : n \in \mathbb{N}\}$ is decreasing with lower bound zero and hence must converge to a fixed point of f (since $p_{n+1} = f(p_n)$). Thus $\lim_n p_n = 0$.

Case 2: If $b = \hat{b}$, then $g(b) = 0$ and $f(p) = p - (4b-1)p^2$, so that $p = 0$ is the unique fixed point of f. Furthermore, $4b - 1 = 3 - 2\sqrt{2} > 0$, so again $f(p) < p$ for all p. It follows again that $\lim_n p_n = 0$.

In these two cases, we can define a Martin-Löf test to prove that $T_d(R) = 0$ for any μ-random closed set R.

For each $m, n \in \mathbb{N}$, let

$$B_m = \{(K, Q) : K_m \cap Q_m \neq \emptyset\},$$

so that $\mu^*(B_m) = p_m$ and let

$$A_{m,n} = \{Q : \mu^*(\{K : K_m \cap Q_m \neq \emptyset\}) \geq 2^{-n}\}.$$

Claim 7.1. For each m and n, $\mu^*(A_{m,n}) \leq 2^n \cdot p_m$.

Proof. Define the Borel measurable function $F_m : \mathcal{C} \times \mathcal{C} \to \{0,1\}$ to be the characteristic function of B_m. Then

$$p_m = \mu^2(B_m) = \int_{Q \in \mathcal{C}} \int_{K \in \mathcal{C}} F(Q,K)dKdQ.$$

Now for fixed Q,

$$\mu^*(\{K : K_m \cap Q_m \neq \emptyset\}) = \int_{K \in \mathcal{C}} F(Q,K)dK,$$

so that for $Q \in A_{m,n}$, we have $\int_{K \in \mathcal{C}} F(Q,K)dK \geq 2^{-n}$. It follows that

$$\begin{aligned} p_m = \int_{Q \in \mathcal{C}} \int_{K \in \mathcal{C}} F(Q,K)dKdQ &\geq \int_{Q \in A_{m,n}} \int_{K \in \mathcal{C}} F(Q,K)dKdQ \\ &\geq \int_{Q \in A_{m,n}} 2^{-n} dQ = 2^{-n}\mu^*(A_{m,n}). \end{aligned}$$

Multiplying both sides by 2^n completes the proof of Claim 7.1. ∎

Since the computable sequence $< p_n >_{n\in\omega}$ converges to 0, there must be a computable subsequence $m_0, m_1, \ldots$ such that $p_{m_n} < 2^{-2n-1}$ for all n. We can now define our Martin-Löf test. Let

$$S_r = A_{m_r,r}$$

and let

$$V_n = \bigcup_{r>n} S_r.$$

It follows that

$$\mu^*(S_r) \leq 2^{r+1}\mu^*(B_{m_r}) < 2^{r+1}2^{-2r-1} = 2^{-r}$$

and therefore

$$\mu^*(V_n) \leq \sum_{r>n} 2^{-r} = 2^{-n}$$

Now suppose that R is a random closed set. The sequence $\langle V_n\rangle_{n\in\omega}$ is a computable sequence of c. e. open sets with measure $\leq 2^{-n}$, so that there is some n such that $R \notin S_n$. Thus for all $r > n$, $\mu^*(\{K : K_{m_r} \cap R_{m_r} \neq \emptyset\}) < 2^{-r}$ and it follows that

$$\mu^*(\{K : K \cap R \neq \emptyset\}) = \lim_n \mu^*(\{K : K_{m_n} \cap R_{m_n} \neq \emptyset\}) = 0.$$

Thus $\mathcal{T}_d(R) = 0$, as desired.

This completes the proof when the function d is independent of σ.

Next suppose that the value b such that $d(\sigma^\frown i) = b \cdot d(\sigma)$ for $i = 0, 1$, depends on σ, say $b_\sigma = d(\sigma^\frown i)/d(\sigma)$ and that $b_\sigma \geq \hat{b}$ for all σ.

Let $f_b(p) = (2b^2 - 4b + 2)p - (1 - 4b + 4b^2)p^2$ as above and let $f_{\hat{b}}(p) = f(p)$. Let p_n be the probability computed above corresponding to $b_\sigma = \hat{b}$ for all σ, so that $p_{n+1} = f(p_n)$. Define p_n^d to be the probability, under μ_d^*, that $K_n \cap Q_n \neq \emptyset$, for closed sets K and Q. We will argue by induction on n that $p_n^d \leq p_n$.

Claim 7.2. For any reals $b, c, p \in [0, 1]$, if $b < c$, then $f_c(p) \leq f_b(p)$.

Proof. Fixing p and taking the derivative of $f_b(p)$ with respect to b, we obtain

$$\frac{\partial f}{\partial b}(b, p) = (4b - 4)p - (8b - 4)p^2 \leq -4bp \leq 0,$$

with the inequality due to the fact that $p^2 \leq p$ on $[0, 1]$. $\square$

Now suppose that for all $\sigma \in \{0, 1\}^*$ and for $i < 2$, $d(\sigma^\frown i) \geq \hat{b}d(\sigma)$ and again let p_n^d be the μ_d-probability that $K_n \cap Q_n \neq \emptyset$. Clearly $p_0^d = 1 = p_0$.

Now assume that $p_n^d \leq p_n$ for any d as above. Let d be given as above with $d((0)) = d((1)) = b \geq \hat{b}$ and define d_i for $i = 0, 1$ as follows.

$$d_i(\sigma^\frown j) = d(i^\frown \sigma^\frown j).$$

Let p^i be the probability under d_i that $Q_n \cap K_n \neq \emptyset$. Then the probability under d_{i+1} that $Q_{n+1} \cap K_{n+1} \neq \emptyset$ can be computed in the four cases as above to equal

$$b^2(p^0 + p^1) + 2b(1 - 2b)(p^0 + p^1) + (1 - 2b)^2(1 - (1 - p^0)(1 - p^1)).$$

By induction, both of p^0 and p^1 are $\leq p_n$ and it follows easily that

$$b^2(p^0 + p^1) + 2b(1 - 2b)(p^0 + p^1) + (1 - 2b)^2(1 - (1 - p^0)(1 - p^1)) \leq f_b(p_n) \leq f(p_n) = p_{n+1}.$$

Finally, suppose that we only have that $b_\sigma \geq \hat{b}$ for σ with $|\sigma| \geq n$. Let R be μ_d^*-random and for each σ of length n, let d_σ be defined so that $d_\sigma(\tau) = d(\sigma^\frown\tau)$ and let $R_\sigma = \{X : \sigma^\frown X \in R\}$. Then R_σ is d_σ-random for each σ, so that the capacity $\mathcal{T}_{d_\sigma}(R_\sigma) = 0$. It follows that $\mathcal{T}_d(R) = 0$ since $Q \cap R \neq \emptyset$ if and only if $Q \cap R \cap I(\sigma) \neq \emptyset$ for some σ of length n.

The appropriate Martin-Löf test can now be given as before to show that any μ_d^*-random closed set will have capacity zero. □

Next we consider the case where random closed sets will have positive capacity.

Theorem 8. *Suppose that $b < \hat{b} = 1 - \frac{\sqrt{2}}{2}$ is fixed and that the measure μ_d is defined by d such that, for all sufficiently long σ, $d(\sigma^\frown 0) = d(\sigma^\frown 1) \leq b \cdot d(\sigma)$. Then $\{R \in \mathcal{C} : \mathcal{T}_d(R) > 0\}$ has μ_d^* measure one and furthermore every μ_d^*-random closed set has positive capacity. Thus for almost all closed sets R, $\mathcal{T}_d(R) > 0$.*

Proof. First fix $b < \hat{b}$ and fix d so that $d(\sigma^\frown i) = d(\sigma) \cdot b$ for all σ and for $i < 2$, and let $\mu^* = \mu_d^*$. Since $0 < 2b^2 - 4b + 1 < 1$, the function $f = f_b$ defined above has a positive fixed point $m_b = \frac{2b^2-4b+1}{(1-2b)^2}$. It is clear that $f(p) > p$ for $0 < p < m_b$ and $f(p) < p$ for $m_b < p$. Furthermore, the function f has its maximum at $p = [\frac{1-b}{1-2b}]^2 > 1$, so that f is monotone increasing on $[0, 1]$ and hence $f(p) > f(m_b) = m_b$ whenever $p > m_b$. As in the proof of Theorem 7 let p_n be the probability that $Q_n \cap K_n \neq \emptyset$ for arbitrary closed sets Q and K. Observe that $p_0 = 1 > m_b$ and hence the sequence $\{p_n : n \in \mathbb{N}\}$ is decreasing with lower bound m_b. It follows that $\lim_n p_n = m_b > 0$.

Now $B = \{(Q, K) : Q \cap K \neq \emptyset\} = \bigcap_n B_n$ is the intersection of a decreasing sequence of sets and hence $\mu^2(B) = \lim_n p_n = m_b > 0$.

Claim 8.1. $\mu^*(\{Q : \mu^*(\{K : K \cap Q \neq \emptyset\}) > 0\}) \geq m_b$.

Proof. Let $B = \{(K, Q) : K \cap Q \neq \emptyset$, let $A = \{Q : \mu^*(\{K : K \cap Q \neq \emptyset\}) > 0\}$ and suppose that $\mu^*(A) < m_b$. As in the proof of Claim 7.1, we have

$$m_b = \mu^2(B) = \int_{Q\in\mathcal{C}} \int_{K\in\mathcal{C}} F(Q, K) dK dQ.$$

For $Q \notin A$, we have $\int_{K\in Q} F(Q, K)dK = \mu^*(\{K : K \cap Q \neq \emptyset\}) = 0$, so that

$$m_b = \int_{Q\in A} \int_{K\in Q} F(Q, K) dK dQ \leq \int_{Q\in A} dQ = \mu^*(A),$$

which completes the proof of Claim 8.1. □

Claim 8.2. $\{Q : \mathcal{T}_d(Q) \geq m_b\}$ has positive measure.

Proof. Recall that $T_d(Q) = \mu^*(\{K : Q \cap K \neq \emptyset\})$. Let $B = \{(K, Q) : K \cap Q \neq \emptyset$, let $A = \{Q : T_d(Q) \geq m_b\}$ and suppose that $\mu^*(A) = 0$. As in the proof of Claim 7.1, we have

$$m_b = \mu^2(B) = \int_{Q\in\mathcal{C}} T_d(Q) dQ.$$

Since $\mu^*(A) = 0$, it follows that for any $B \subseteq \mathcal{C}$, we have

$$\int_{Q\in B} T_d(Q) dQ \leq m_b \mu^*(B).$$

Furthermore, $T_d(Q) < m_b$ for almost all Q, so there exists some P with $T_d(P) < m_b - \epsilon$ for some positive ϵ. This means that for some n, $\mu^*(\{K : P_n \cap K_n \neq \emptyset\}) < m_b - \epsilon$. Then for *any* closed set Q with $Q_n = P_n$, we have $T_d(Q) < m_b - \epsilon$. But $E = \{Q : Q_n = P_n\}$ has positive measure, say $\delta > 0$. Then we have

$$\begin{aligned} m_b = \int_{Q\in\mathcal{C}} T_d(Q)dQ &= \int_{Q\in E} T_d(Q)dQ \; + \; \int_{Q\notin E} T_d(Q)dQ \\ &\leq \; \delta(m_b - \epsilon) + (1-\delta)m_b = m_b - \epsilon\delta < m_b. \end{aligned}$$

This contradiction demonstrates Claim 8.2. □

It is now easy to see that $\mathcal{T}_d(R) > 0$ with probability one. That is, let p be the probability that $\mathcal{T}_d(R) = 0$. Then by considering the first level of R, we can see that $p = 2bp + (1-2b)p^2$ and hence either $p = 0$ or $p = 1$. Since we know that $p < 1$, it follows that $p = 0$.

Since the set of μ^*-random closed sets has measure one, there must be a random closed set R such that $\mathcal{T}_d(R) \geq m_b$ and furthermore, almost every μ^*-random closed set has positive capacity.

Furthermore, we can construct a Martin-Löf test as follows. First observe that for any computable q, $\{Q : \mathcal{T}_d(Q) < q\}$ is a c. e. open set. This is because $\mathcal{T}_d(Q) < q \iff (\exists n)\mathcal{T}_d(Q_n) < q$ and $\mathcal{T}_d(Q_n)$ can be uniformly computed from Q.

Now let $h(p)$ be the probability that $\mathcal{T}_d(Q) < p$. Note that if $\mathcal{T}_d(Q_i) \geq p$ for $i = 0$ or for $i = 1$, then $\mathcal{T}_d(Q) \geq bp$. It follows that $h(bp) \leq h(p)^2$. Since $\mathcal{T}_d(Q) = 0$ with probability zero, it follows that $\lim_{p\to 0} h(p) = 0$. Take a rational q small enough so that $h(q) < \frac{1}{2}$. Then $h(b^n q) \leq (\frac{1}{2})^{2^n} \leq 2^{-n}$. Let $S_n = \{Q : \mathcal{T}_d(Q) \leq b^n q\}$. Then $\mu_d^*(S_n) \leq 2^{-n}$ and the sequence (S_n) is effectively c. e. open, so that no random closed set can be belong to all S_n. But if $\mathcal{T}_d(Q) = 0$, then of course $Q \in S_n$ for all n. Thus every μ_d^* random closed set must have positive capacity.

This completes the proof when d is independent of σ.

Next suppose that $b < \hat{b}$ and that, for all σ, $d(\sigma^\frown 0) = d(\sigma^\frown 1) \leq b \cdot d(\sigma)$. Let p_n^d now be the μ_d^* probability that $Q_n \cap K_n \neq \emptyset$. It follows from the monotonicity of f (Claim 3) that $p_n^d \geq p_n$ for all d as above and thus $\lim_n p_n^d \geq m_b$. The same argument as above now shows that $\{Q : \mathcal{T}_d(Q) \geq m_b\}$ has positive measure and thus $\mathcal{T}_d(Q)$ has positive capacity with probability one. The argument that every μ_d^*-random closed set has positive capacity follows as above. □

Note that random closed sets can have arbitrarily small positive capacity. This follows from the fact that $\mathcal{T}_d(0^\frown Q) = (1-b)\mathcal{T}_d(Q)$.

Thus for certain measures, there exists a random closed set with measure zero but with positive capacity. For the standard measure, a random closed set has capacity zero.

Corollary 9. *Let d be the uniform measure with $b_0 = b_1 = b_2 = \frac{1}{3}$. Then for any μ_d^*-random closed set R, $\mathcal{T}_d(R) = 0$.* □

Finally, we consider non-symmetric measures, where $d(\sigma^\frown 0)$ does not necessarily equal $d(\sigma^\frown 1)$. We will give the result where μ_d is a uniform measure. The proofs follow the same outline as those of Theorems 7 and 8.

Theorem 10. *Fix b and let μ_d be a measure defined by d where $d(\sigma^\frown i) = b_i \cdot d(\sigma)$ with $b_0 + b_1 = 2b > 0$ and $b_2 = 1 - 2b > 0$ and let $\hat{b} = 1 - \frac{\sqrt{2}}{2}$. Then*

(1) If $b \geq \hat{b}$ and $|b_0 - b_1| \leq \sqrt{8b - 4b^2 - 2}$, then for any μ_d^*-random closed set R, $\mathcal{T}_d(R) = 0$. Thus for almost all closed sets R, $\mathcal{T}_d(R) = 0$.
(2) If $b > \hat{b}$ or $|b_0 - b_1| > \sqrt{8b - 4b^2 - 2}$, then there is a μ_d^*-random closed set R with $\mathcal{T}_d(R) > 0$.

Proof. For convenience let $\mu = \mu_d^*$ and let $\mu^2 = \mu \times \mu$ be the usual product measure on the product space $\mathcal{C} \times \mathcal{C}$. We will compute the probability $p = \mu^2(\{(Q, K) : Q \cap K \neq \emptyset\})$.

As in the proof of Theorem 7 let p_n be the probability that $Q_n \cap K_n \neq \emptyset$ for arbitrary closed sets Q and K, so that $p = \lim_n p_n$ Clearly, $p_1 = 1 - 2b_0 b_1$ since $Q_1 \cap K_1 = \emptyset$ only when $Q_1 = I(i)$ and $K_1 = I(1-i)$. We will compute as before a quadratic function f so that $p_{n+1} = f(p_n)$. Considering the various cases as in the proof of Theorem 7, we see that

$$\begin{aligned} p_{n+1} &= (b_0^2 + b_1^2 + 4b(1-2b))p_n + (1-2b)^2(2p_n - p_n^2) \\ &= (2b_0 - 4bb_0 + 4b^2 + 4b + 2)p_n - (1-2b)^2 p_n^2 \end{aligned}$$

Next, we investigate $\lim_n p_n$. Let

$$f(p) = (2b_0 - 4bb_0 + 4b^2 + 4b + 2)p - (1-2b)^2 p^2$$

This function has fixed points $p = 0$ and $p = \frac{2b_0 - 4bb_0 + 4b^2 + 4b + 1}{(1-2b)^2}$. Note that we must have $b < \frac{1}{2}$ so $(1-2b)^2 > 0$.

Now consider the functions $g(a) = 2a - 4ba + 4b^2 + 4b + 1$, which has roots $a_\pm = b \pm \sqrt{-b^2 + 2b - \frac{1}{2}}$ and $h(b) = -b^2 + 2b - \frac{1}{2} = -2\big(2(b-1)^2 - 1\big)$, which has root $\hat{b}$. There are 3 cases to consider when comparing b and $\hat{b}$.

(1) If $b > \hat{b}$ and $a_- \leq b_0 \leq a_+$, then $g(b_0) < 0$ and hence the nonzero fixed point of f is negative. Since (p_n) is decreasing with lower bound 0 the sequence converges to a non-negative fixed point of f. Hence $p = \lim_n p_n = 0$.
(2) If $b = \hat{b}$ and $b_0 = b$ or if $b_0 = a_\pm$ then $g(b_0) = 0$ and so $p = 0$ is the only fixed point of f hence $p = \lim_n p_n = 0$.
(3) If $b < \hat{b}$ or $b_0 \notin [a_-, a_+]$, then $g(b_0) > 0$ and so f has positive fixed point $m_{b,b_0} = \frac{2b_0 - 4bb_0 + 4b^2 + 4b + 1}{(1-2b)^2}$. Furthermore, f has its maximum at $p = \frac{b_0 - 2bb_0 + 2b^2 + 2b + 1}{(1-2b)^2} > 1$ (since $2b > 2bb_0$). Thus f is increasing for $p < 1$, so if $p > m_{b,b_0}$, then $f(p) > f(m_{b,b_0}) = m_{b,b_0}$. Hence, since $p_0 = 1$, (p_n) is bounded below by $m_{b,b}$ and so, $p = \lim_n p_n = m_{b,b_0} > 0$. □

Due to he inequalities needed for $|b_0 - b_1|$ in the theorem, it seems that the proof given above does not easily extend to provide a result for non-uniform measures or to prove that, in the second case above, *every* random closed set has positive capacity.

4. Effectively Closed Sets

In this section, we consider the capacity of effectively closed sets. A random closed set can never be effectively closed. But we can still construct an effectively closed set with measure zero and with positive capacity.

We begin by characterizing the possible capacity of effectively closed sets. For the following results we will take $\mathcal{T} = \mathcal{T}_d$ where μ_d is the computable measure defined by $d(\sigma^\frown i) = b_i$ with $0 < b_1 \leq b_0$ and $1 > b_0 + b_1 > 0$. For any effectively closed set

$Q = [T]$, Q is the effective intersection of the decreasing sequence $[T_n]$ of clopen sets, where $T_n = T \cap \{0,1\}^{\leq n}$. Thus for a computable measure $\mathcal{T}_d$, the capacity $\mathcal{T}_d(Q)$ is the limit of a computable, decreasing sequence and is therefore an upper semi-computable real. We will show that for *every* upper semi-computable real $q \in [0,1]$, there exists an effectively closed set Q with $\mathcal{T}_d(Q) = q$.

Lemma 11. *Let $Q = 0^\frown Q_0 \cup 1^\frown Q_1$ and let $q_i = \mathcal{T}(Q_i)$ for $i \leq 1$. Then, $\mathcal{T}(Q) = (1-b_1)q_0 + (1-b_0)q_1 - (1-(b_0+b_1)) \cdot q_0 q_1$.*

Proof. For a closed set K, $K \cap Q \neq \emptyset$ if and only if one of the following holds:

1. $K = 0^\frown K_0$ and $Q_0 \cap K_0 \neq \emptyset$ (which has probability $b_0 \cdot \mathcal{T}(Q_0)$), or
2. $K = 1^\frown K_1$ and $Q_1 \cap K_1 \neq \emptyset$ (which has probability $b_1 \cdot \mathcal{T}(Q_1)$), or
3. $K = 0^\frown K_0 \cup 1^\frown K_1$ and either $Q_0 \cap K_0 \neq \emptyset$ or $Q_1 \cap K_1 \neq \emptyset$ (which has probability $(1-(b_0+b_1))(1-(1-\mathcal{T}(Q_0)(1-\mathcal{T}(Q_1)))$.

Thus,

$$\begin{aligned}\mathcal{T}(Q) &= b_0 q_0 + b_1 q_1 + (1-(b_0+b_1))(1-(1-q_0)(1-q_1)) \\ &= (1-b_1)q_0 + (1-b_0)q_1 - (1-(b_0+b_1))q_0 q_1\end{aligned}$$

□

Lemma 12. *Let $Q = \bigcup_{k=0}^{k=n} I(\sigma_k)$. Then for each $j \leq k$, $\mathcal{T}(Q) - \mathcal{T}(Q \setminus I(\sigma_j)) \leq (1-b_1)^{|\sigma_j|}$.*

Proof. The proof is by induction on $|\sigma_j|$. If $|\sigma_j| = 0$, this is trivial.

Let $Q = 0^\frown Q_0 \cup 1^\frown Q_1$ and let $q_i = \mathcal{T}(Q_i)$ for $i = 0, 1$. If $\sigma_i = (i)$, then $\mathcal{T}(Q) = (1-b_{1-i}) + b_{1-i} \cdot q_i$ and $\mathcal{T}(Q \setminus I(i)) = (1-b_i) \cdot q_{1-i}$. Thus, $\mathcal{T}(Q) - \mathcal{T}(Q \setminus I(i)) = (1-b_{1-i}) - (1-(b_0+b_1)) \cdot q_i \leq (1-b_1$.

Now let $|\sigma_j| = n > 0$ and let $\sigma_j = i^\frown \tau$ for some $i \leq 1$ and some τ. Let $r = \mathcal{T}(Q_i \setminus I(\tau))$. Then, $\mathcal{T}(Q) - \mathcal{T}(Q \setminus I(\sigma_j)) = (1-b_{1-i})(q_i - r) - (1-(b_0+b_1)q_{1-i}(q-r) \leq (1-b_1)(q-r) \leq (1-b_1)(1-b_1)^{n-1}$, where the last inequality holds by the induction hypothesis. □

Theorem 13. *Let the real number $q \in [0,1]$ be upper semi-computable, i.e. there is a computable, decreasing sequence $\{q_n : n \in \mathbb{N}\}$ such that $\lim q_n = q$. Then there exists an effectively closed set P such that $\mathcal{T}(P) = q$. Moreover, P can be written as $\bigcap_n P_n$ where $\{P_n : n \in \mathbb{N}$ is a computable sequence of clopen sets with $q_{n+1} \leq \mathcal{T}(P_n) \leq q_n$.*

Proof. We may assume without loss of generality that $q_0 = 1$. We will construct P_n by recursion beginning with $P_0 = 2^{\mathbb{N}}$. Now suppose we have constructed the clopen set $Q_{n-1} = \bigcup_{k=0}^{m} I(\sigma_k)$ such that $q_n \leq \mathcal{T}_d(Q_{n-1}) \leq q_{n-1}$.

Let $\delta = q_n - q_{n-1}$ and compute s large enough so that $(1-b_1)^s < \delta$ and so that $|\sigma_k| \leq s$ for all $k \leq m$. Then we can rewrite each interval $I(\sigma_k)$ as a union of intervals $I(\tau)$ with $|\sigma| = s$ and thus obtain $Q_{n-1} = \bigcup_{k=0}^{r} I(\tau_k)$ with $|\tau_k| = s$ for all $k \leq r$. Now let $Q_{n-1,k} = \bigcup_{j=0}^{k-1} I(\tau_j)$ for each $k \leq r+1$, so that $Q_{n-1,k} = Q_{n-1,k+1} \setminus I(\tau_k)$ for each $k \leq r$. Observe that $\mathcal{T}_d(Q_{n-1,r+1}) = \mathcal{T}_d(Q_{n-1}) \geq q_n$ and that $\mathcal{T}_d(Q_{n-1,0}) = \mathcal{T}_d(\emptyset) = 0 \leq q_n$.

It follows from Lemma 12 that, for any k, $\mathcal{T}_d(Q_{n-1,k+1}) - \mathcal{T}_d(Q_{n-1,k}) \leq \delta$. Now let k be the least such that $\mathcal{T}_d(Q_{n-1,k}) \leq q_n$. Then $\mathcal{T}_d(Q_{n-1,k+1}) > q_n$ and also $\mathcal{T}_d(Q_{n-1,k+1}) \leq \mathcal{T}_d(Q_{n-1,k}) + \delta \leq q_n + \delta \leq q_{n-1}$. So we let $Q_n = Q_{n-1,k+1}$.

In this way, we have constructed a computable, decreasing sequence Q_n of clopen sets with $q_n \leq \mathcal{T}_d(Q_n) \leq q_{n-1}$, so that, for $Q = \bigcap_n Q_n$, we have $\mathcal{T}_d(Q) = \lim_n \mathcal{T}_d(Q_n) = q$. □

Theorem 14. *For the uniform measure μ_d defined by $d(\sigma^\frown i) = b \cdot d(\sigma)$ for all σ, there is an effectively closed set Q with Lebesgue measure zero and positive capacity $\mathcal{T}_d(Q)$.*

Proof. First let us compute the capacity of $X_n = \{x : x(n) = 0\}$. For $n = 0$, we have $\mathcal{T}_d(X_0) = 1 - b$. That is, Q meets X_0 if and only if $Q_0 = I(0)$ (which occurs with probability b), or $Q_0 = 2^{\mathbb{N}}$ (which occurs with probability $1 - 2b$). Now the probability $\mathcal{T}_d(X_{n+1})$ that an arbitrary closed set K meets X_{n+1} may be calculated in two distinct cases. As in the proof of Theorem 9, let

$$K_n = \bigcup\{I(\sigma) : \sigma \in \{0,1\}^n \ \& \ K \cap I(\sigma) \neq \emptyset\}$$

Case I If $K_0 = 2^{\mathbb{N}}$, then $\mathcal{T}_d(X_{n+1}) = 1 - (1 - \mathcal{T}_d(X_n))^2$.

Case II If $K_0 = I((i))$ for some $i < 2$, then $\mathcal{T}_d(X_{n+1}) = \mathcal{T}_d(X_n)$.

It follows that

$$\begin{aligned}\mathcal{T}_d(X_{n+1}) &= 2b \cdot \mathcal{T}_d(X_n) + (1 - 2b)(2\mathcal{T}_d(X_n) - (\mathcal{T}_d(X_n))^2) \\ &= (2 - 2b)\mathcal{T}_d(X_n) - (1 - 2b)(\mathcal{T}_d(X_n))^2\end{aligned}$$

Now consider the function $f(p) = (2 - 2b)p - (1 - 2b)p^2$, where $0 < b < \frac{1}{2}$. This function has the properties that $f(0) = 0$, $f(1) = 1$ and $f(p) > p$ for $0 < p < 1$. Since $\mathcal{T}_d(X_{n+1}) = f(\mathcal{T}_d(X_n))$, it follows that $\lim_n \mathcal{T}_d(X_n) = 1$ and is the limit of a computable sequence.

For any $\sigma = (n_0, n_1, \ldots, n_k) \in \mathbb{N}^{\mathbb{N}}$, with $n_0 < n_1 < \cdots < n_k$, similarly define $X_\sigma = \{x : (\forall i \leq k) x(n_i) = 0\}$. A similar argument to that above shows that $\lim_n \mathcal{T}_d(X_{\sigma^\frown n})/\mathcal{T}_d(X_\sigma) = 1$.

Now consider the decreasing sequence $c_k = \frac{2^{k+1}+1}{2^{k+2}}$ with limit $\frac{1}{2}$. Choose $n = n_0$ such that $\mathcal{T}_d(X_n) \geq \frac{3}{4} = c_0$ and for each k, choose $n = n_{k+1}$ such that $\mathcal{T}_d(X_{(n_0,\ldots,n_k,n)}) \geq c_{k+1}$. This can be done since $c_{k+1} < c_k$. Finally, let $Q = \bigcap_k X_{(n_0,\ldots,n_k)}$. Then $\mathcal{T}_d(Q) = \lim_k \mathcal{T}_d(X_{(n_0,\ldots,n_k)}) \geq \lim_k c_k = \frac{1}{2}$. ∎

It is clear that we can make the capacity in Theorem 14 arbitrarily large below 1.

5. Conclusions

In this paper, we have established a connection between measure and capacity for the space $\mathcal{C}$ of closed subsets of $2^{\mathbb{N}}$. We showed that for a computable measure μ^*, a computable capacity may be defined by letting $\mathcal{T}(Q)$ be the measure of the family of closed sets K which have nonempty intersection with Q. We have proved an effective version of the Choquet's theorem by showing that every computable capacity may be obtained from a computable measure in this way.

We have established conditions on computable measures that characterize when the capacity of a random closed set equals zero or is > 0. In particular, for symmetric measures where $d(\sigma^\frown 0) = d(\sigma^\frown 1) = b \cdot d(\sigma)$ for all σ, where b depends on σ, we have shown the following. If $d(\sigma^\frown 2) \leq \frac{\sqrt{2}}{2} d(\sigma)$ for all σ, then $\mathcal{T}_d(R) = 0$ for any μ_d^*-random closed set R. If $d(\sigma^\frown 2) \geq b \cdot d(\sigma)$ for all σ, where $b > \frac{\sqrt{2}}{2}$ is a constant, then $\mathcal{T}_d(R) > 0$ for any μ_d^*-random closed set R.

We have shown that the set of capacities of an effectively closed set is exactly the set of upper semi-computable reals. We have also constructed effectively closed set with positive capacity and with Lebesgue measure zero.

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