Computability of Countable Subshifts in One Dimension *

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Abstract

We investigate the computability of countable subshifts in one dimension, and their members. Subshifts of Cantor-Bendixson rank two contain only eventually periodic elements. Any rank two subshift in $2^{\mathbb{Z}}$ is is decidable. Subshifts of rank three may contain members of arbitrary Turing degree. In contrast, effectively closed (Π^0_1) subshifts of rank three contain only computable elements, but Π^0_1 subshifts of rank four may contain members of arbitrary Δ^0_2 degree. There is no subshift of rank $\omega + 1$.

Keywords: Computability, Symbolic Dynamics, Π^0_1 Classes

1 Introduction

There is a long history of interaction between computability and dynamical systems. A Turing machine may be viewed as a dynamical system which produces a sequence of configurations or words before possibly halting. The reverse notion of using an arbitrary dynamical system for general computation has generated much interesting work. See for example [1, 12]. In this paper we will consider computable aspects of certain dynamical systems over the Cantor space $2^{\mathbb{N}}$ and the related space $2^{\mathbb{Z}}$.

The study of computable dynamical systems is part of the Nerode program to study the effective content of theorems and constructions in analysis. Weihrauch

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[23] has provided a comprehensive foundation for computability theory on various spaces, including the space of compact sets and the space of continuous real functions.

Computable analysis is related as well to the so-called reverse mathematics of Friedman and Simpson [20], where one studies the proof-theoretic content of various mathematical results. The study of reverse mathematics is related in turn to the concept of *degrees of difficulty.* Here we say that $P \leq_M Q$ if there is a Turing computable functional F which maps Q into P; thus the problem of finding an element of P can be uniformly reduced to that of finding an element of Q, so that P is *less difficult than* Q. See Medvedev [16] and Sorbi [22] for details. The degrees of difficulty of effectively closed sets (also known as Π^0_1 classes) have been intensively investigated in several recent papers, for example Cenzer and Hinman [9] and Simpson [19].

The computability of Julia sets in the reals has been studied by Cenzer [3] and Ko [14]. The computability of complex dynamical systems has been investigated by Rettinger and Weihrauch [18] and by Braverman and Yampolsky [2]. The study of the computability of dynamical systems has received increasing attention in recent years; see for example papers of Delvenne et al [12], Hochman [13], Miller [17] and Simpson [21].

The connection between dynamical systems and subshifts is the following. Certain dynamical systems may be given by a continuous function F on a *symbolic space* $\mathcal{X}$ (one with a basis of clopen sets). For each $X \in \mathcal{X}$, the sequence $(X, F(X), F(F(X)), \dots)$ is the *trajectory* of X. Given a fixed partition $U_0, \dots, U_{k-1}$ of $\mathcal{X}$ into clopen sets, the *itinerary* $It(X)$ of a point X is the sequence $(a_0, a_1, \dots) \in k^{\mathbb{N}}$ where $a_n = i$ iff $F^n(X) \in U_i$. Let $It[F] = \{It(X) : X \in \mathcal{X}\}$. Note that $It[F]$ will be a closed set. We observe that, for each point X with itinerary $(a_0, a_1, \dots)$, the point $F(X)$ has itinerary $(a_1, a_2, \dots)$. Now the shift operator σ on $k^{\mathbb{N}}$ is defined by $\sigma(a_0, a_1, \dots) = (a_1, a_2, \dots)$. It follows that $It[F]$ is closed under the shift operator, that is, $It[F]$ is a *subshift.*

Computable subshifts and the connection with effective symbolic dynamics were investigated by Cenzer, Dashti and King [6] in a recent paper. A total, Turing computable functional $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is always continuous and thus will be termed *computably continuous* or just computable. Effectively closed sets (also known as Π^0_1 classes) are a central topic in computability theory; see [10] and Section 2 below. It was shown for any computably continuous function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, $It[F]$ is a decidable Π^0_1 class and, conversely, any decidable Π^0_1 subshift P is $It[F]$ for some computable map F. In this paper, Π^0_1 subshifts are constructed in $2^{\mathbb{N}}$ and in $2^{\mathbb{Z}}$ which have no computable elements and are not decidable. Thus there is a Π^0_1 subshift with non-trivial Medvedev degree. J. Miller [17] showed that *every* Π^0_1 Medvedev degree contains a Π^0_1 subshift. Simpson [21] studied Π^0_1 subshifts in two dimensions and showed that every Π^0_1 Medvedev degree contains a Π^0_1 subshift of finite type which is a stronger result than just containing a Π^0_1 subshift.

Now every nonempty countable Π^0_1 class contains a computable element, so that all countable Π^0_1 classes have Medvedev degree $\mathbf{0}$, and many uncountable classes also have degree $\mathbf{0}$. In the present paper, we will consider more closely the structure of countable subshifts, using the Cantor-Bendixson (CB) derivative. We will compare and contrast countable subshifts of finite CB rank with Π^0_1 subshifts of finite CB rank as well as with arbitrary Π^0_1 classes of finite rank.

This paper is an extended and significantly revised version of the conference paper [7]. The original paper deals only with subshifts in $2^{\mathbb{N}}$, whereas this new paper also deals with subshifts in $2^{\mathbb{Z}}$, where some of the results turn out to be different. For example, Theorem 3.13 below says that there are subshifts in $2^{\mathbb{N}}$ of rank two of arbitrary Turing degree, the new Theorems 3.14 (with Corollary 3.15) and 4.1 say that in $2^{\mathbb{Z}}$, this fails for subshifts of rank two, but holds for subshifts of rank three. The new results Lemmas 2.4, and 3.5 connect subshifts of $2^{\mathbb{N}}$ with subshifts of $2^{\mathbb{Z}}$ in terms of the interesting Hessenberg sum of ordinals and the new Corollary 4.7 relates the degrees of the members of subshifts in $2^{\mathbb{N}}$ with those in $2^{\mathbb{Z}}$. The entirely new Theorem 4.6 relates arbitrary closed sets and Π^0_1 classes with subshifts and Π^0_1 subshifts and leads to a greatly improved version of Theorem 8 of [7].

The outline of this paper is as follows. Section 2 contains definitions and preliminaries. Here we define the *join* $X^-.Y \in 2^{\mathbb{Z}}$ of two elements of $2^{\mathbb{N}}$ and the product $P \otimes Q = \{X^-.Y : X \in P \;\&\; Y \in Q\} \subseteq 2^{\mathbb{Z}}$ of subsets P and Q of $2^{\mathbb{N}}$, and prove that the CB rank of $X^-.Y$ in $P \otimes Q$ equals the *Hessenberg sum* of the rank of X in P and the rank of Y in Q.

Section 3 focuses on subshifts of rank two and has some general results about periodic and eventually periodic members of subshifts. We show that if Q is a subshift of rank two, then every member of Q is eventually periodic (and therefore computable) and furthermore if $Q \subseteq 2^{\mathbb{Z}}$, then the members of rank two are periodic and Q is a decidable closed set. However, there are rank two subshifts in $2^{\mathbb{N}}$ of arbitrary Turing degree and rank two Π^0_1 subshifts of arbitrary c. e. degree, so that rank two undecidable Π^0_1 subshifts exist in $2^{\mathbb{N}}$. We give conditions under which a rank two subshift in $2^{\mathbb{N}}$ must be decidable. We show that there is no subshift of rank $\omega + 1$ and give an example of a subshift of rank $\omega + 2$.

In section 4, we study subshifts of rank three and four. We show that subshifts of rank three may contain members of arbitrary Turing degree and that subshifts of rank three in $2^{\mathbb{Z}}$ may have arbitrary Turing degree. In contrast, we show that Π^0_1 subshifts of rank three contain only computable elements, but Π^0_1 subshifts of rank four may contain members of arbitrary c. e. degree. More generally, we show that for any given Π^0_1 class P of rank two, there is a subshift Q of rank four such that the degrees of the members of P and the degrees of the members of Q are identical. We prove that if P is a closed subset of $2^{\mathbb{N}}$ of CB rank α, then there is a subshift $Q \subseteq 2^{\mathbb{N}}$ of rank $\alpha+2$, a computable injection from P into Q and a countable-to-one degree-preserving mapping from $Q \setminus D^{\alpha+1}(Q)$ onto P; furthermore $D^{\alpha+1}(Q)$ is the set of eventually periodic points of Q. There is also a subshift Q_1 of $2^{\mathbb{Z}}$ with similar mappings. This implies, for $\alpha = 2$, that for any degree $\mathbf{b}$ such that either $\mathbf{b} \leq_T \mathbf{0}'$ or $\mathbf{0}' \leq_T \mathbf{b} \leq_T \mathbf{0}''$, there is a Π^0_1 subshift $Q \subseteq 2^{\mathbb{N}}$ of rank four (and also a Π^0_1 subshift $Q_1 \subseteq 2^{\mathbb{Z}}$) such that

(i) Every element of Q (Q_1) of rank 2 or 3 is eventually periodic.

(ii) Every element of Q (Q_1) of rank 1 has Turing degree $\mathbf{b}$.

It is also shown that for any Π^0_1 class $P \subseteq 2^{\mathbb{N}}$, there is a Π^0_1 subshift $Q \subseteq 2^{\mathbb{N}}$ such that the non-computable degrees of the members of Q are identical with the non-computable degrees of the member of P.

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2 Preliminaries

We begin with some basic definitions. Let $\mathbb{N} = \{0, 1, 2, ...\}$ denote the set of natural numbers. For any set Σ and any $i \in \mathbb{N}$, Σ^i denotes the strings of length i from Σ, Σ^* denotes the set of all finite strings from Σ. $\Sigma^{\mathbb{N}}$ denotes the set of countably infinite sequences from Σ, that is, the set of functions mapping $\mathbb{N}$ into Σ, and $\Sigma^{\mathbb{Z}}$ denotes the set of functions mapping $\mathbb{Z}$ into Σ. We write $\Sigma^{<n}$ for $\bigcup_{i<n} \Sigma^i$. For any set A, we let card(A) denote the cardinality of the set A.

For a string $w = (w(0), w(1), \ldots, w(n-1))$, $|w|$ denotes the length n of w. The *reverse* of a string $w = (w(0), \ldots, w(n-1))$ is the string $w^- = (w(n-1), \ldots, w(0))$. The shift function on strings is defined by $\sigma(w) = (w(1), \ldots, w(|w|-1))$. For $X \in 2^{\mathbb{N}}$, $\sigma(X) = (X(1), X(2), \ldots)$–the result of deleting the initial entry of X. For $Z \in 2^{\mathbb{Z}}$, $Y = \sigma(Z)$ is defined so that $Y(i) = Z(i+1)$. The empty string has length 0 and will be denoted by λ. A length n string of k's will be denoted k^n. For $m < |w|$, $w \upharpoonright m$ is the string $(w(0), \ldots, w(m-1))$.

Given two strings v and w, the *concatenation* $v^\frown w$ is defined by

$$v^\frown w = (v(0), v(1), \ldots, v(m-1), w(0), w(1), \ldots, w(n-1)),$$

where $|v| = m$ and $|w| = n$. For $a \in \Sigma$, we write $w^\frown a$ (or just wa) for $w^\frown(a)$ and we write $a^\frown w$ (or just aw) for $(a)^\frown w$. We say w is an *initial segment* or *prefix* of v (written $w \preceq v$) if $v = w^\frown x$ for some x; this is equivalent to saying that $w = v \upharpoonright m$ for some m; w is a *suffix* of v if $v = x^\frown w$ for some x; w is a *factor* of v if $v = x^\frown w^\frown y$ for some x and y.

For any $X \in \Sigma^{\mathbb{N}}$ and any finite n, the *initial segment* $X \upharpoonright n$ is $(X(0), \ldots, X(n-1))$. For $Z \in \Sigma^{\mathbb{Z}}$ and $i \leq j$ from $\mathbb{Z}$, $Z[i,j]$ denotes the finite string $(Z(i), Z(i+1), \ldots, Z(j))$; $Z[i,j)$, $Z(i,j]$, and $Z(i,j)$ are similarly defined. These definitions also apply to $Z \in \Sigma^{\mathbb{N}}$ as well as to finite strings. We say that a word v is a *factor* of $X \in \Sigma^{\mathbb{N}}$ or of $X \in \Sigma^{\mathbb{Z}}$ if $v = X[i,j]$ for some i and j.

For a string $w \in \Sigma^*$ and any $X \in \Sigma^{\mathbb{N}}$, we write $w \prec X$ if $w = X \upharpoonright n$ for some n. For any $w \in \Sigma^n$ and any $X \in \Sigma^{\mathbb{N}}$, we let $w^\frown X = (w(0), \ldots, w(n-1), X(0), X(1), \ldots)$. For $X \in \Sigma^{\mathbb{N}}$, let $X^- = (\cdots, X(2), X(1), X(0))$; that is, for each $n \in \mathbb{Z}$, let $X^-(-i-1) = on.X(i)$, so that $X^- \in \Sigma^{Z^-}$, where $Z^- = \{i \in Z : i < 0\}$. For $X, Y \in \Sigma^{\mathbb{N}}$, let $Z = X^-.Y \in \Sigma^{\mathbb{Z}}$ be defined so that, for all $n \in \mathbb{N}$, $Z(n) = Y(n)$ and $Z(-n-1) = X(n)$. We denote by v^ω the infinite concatenation $v^\frown v^\frown \cdots$; similarly $v^{-\omega} = \cdots v^\frown v$ and $v^\infty = v^{-\omega}.v^\omega$.

The topology on $2^{\mathbb{N}}$ has a basis of *intervals*, which are clopen sets of the form

$$[w] = \{X : w \prec X\}.$$

A subset of $2^{\mathbb{N}}$ is clopen if and only if it is a finite union of basic intervals. The topology on $2^{\mathbb{Z}}$ has a basis of clopen sets of the form $\{Z \in 2^{\mathbb{Z}} : Z[-n,n] = w\}$, where w has odd length.

A *tree* T over Σ^* is a set of finite strings from Σ^* which contains the empty string λ and which is closed under initial segments. We say that $w \in T$ is an *immediate successor* of $v \in T$ if $w = va$ for some $a \in \Sigma$. We will assume that $\Sigma \subseteq \mathbb{N}$, so that $T \subseteq \mathbb{N}^*$. A *bi-tree* T is a set of finite strings of odd length which is closed under central segments, that is, $(x(-n-1), x(-n), \dots, x(0), x(1), \dots, x(n), x(n+1)) \in T$ implies that $(x(-n), x(-n+1), \dots, x(0), x(1), \dots, x(n-1), x(n)) \in T$.

For any tree T, $(X(0), X(1), \dots)$ is said to be an infinite path through T if $X \upharpoonright n \in T$ for all n. We let $[T]$ denote the set of infinite paths through T. It is well-known that a subset Q of $2^{\mathbb{N}}$ is closed if and only if $Q = [T]$ for some tree T. A subset P of $2^{\mathbb{N}}$ is a Π^0_1 *class* (or *effectively closed set*) if $P = [T]$ for some computable tree T. For any closed set P, define the tree T_P to be $\{w \in \mathbb{N}^*$ such that $P \cap [w] \neq \emptyset\}$. For any tree T, we say that a node $w \in T$ is *extendible* if there exists $X \in [T]$ such that $w \prec X$. If $P = [T]$, then T_P will equal the set of extendible nodes of T and will not depend on T. If T is computable, then the set of extendible nodes is a tree which is a co-c. e. subset of Σ^* but is not in general computable. P is said to be *decidable* (or *computable*) if T_P is a computable set.

For a bi-tree T, $Z \in 2^{\mathbb{Z}}$ is a bi-infinite path through T if $Z[-n,n] \in T$ for all n. Here also $[T]$ is the set of bi-infinite paths through T and $Q \subseteq 2^{\mathbb{Z}}$ is closed if and only if $Q = [T]$ for some bi-tree T. The definitions of effectively closed sets, extendible nodes, and decidable closed is analogous to those given above for $2^{\mathbb{N}}$.

For two closed sets P and Q in $2^{\mathbb{N}}$, let $P \otimes Q = \{X^-.Y : X \in P \;\&\; Y \in Q\}$, which will be a closed set in $2^{\mathbb{Z}}$. Note that our version of $P \otimes Q$ is computably homeomorphic to the usual definition (see [4]) as the set of sequences $X \oplus Y = (X(0), Y(0), X(1), Y(1), \dots)$ for $X \in P$ and $Y \in Q$.

The closed set P is subsimilar (or a subshift) if T_P is subsimilar. (Thus being closed is a part of our definition of a subshift.) A tree $T \subseteq 2^{\mathbb{N}}$ is said to be *subsimilar* if for every v and w, $vw \in T$ implies $w \in T$. A bi-tree T is said to be subsimilar if for every v of even length and every w, $vw \in T$ or $wv \in T$ implies that $w \in T$.

We say that a set Q *avoids* a word w if w is not a factor of any $X \in Q$. For any subset S of $\{0,1\}^*$, let $Q^{\mathbb{Z}}_S = \{X \in 2^{\mathbb{Z}} : X$ avoids w for all $w \in S\}$ and let $Q^{\mathbb{N}}_S = \{X \in 2^{\mathbb{N}} : X$ avoids w for all $w \in S\}$.

The following well-known result is useful in showing that a given set Q is in fact a subshift. See Proposition 2 of [6] for a proof.

Proposition 2.1. *For any $S \subseteq \{0,1\}^*$, $Q^{\mathbb{Z}}_S$ is a subshift in $2^{\mathbb{Z}}$ and $Q^{\mathbb{N}}_S$ is a subshift in $2^{\mathbb{N}}$. Furthermore, if S is a c. e. set, then the sets $Q^{\mathbb{Z}}_S$ and $Q^{\mathbb{N}}_S$ are Π^0_1 classes.*

This proposition has a partial converse.

Proposition 2.2. *If the set Q in $2^{\mathbb{N}}$ (or $2^{\mathbb{Z}}$) is a subshift, then there is a set S of words such that $Q = Q_S^{\mathbb{N}}$ (or $Q = Q_S^{\mathbb{Z}}$). Furthermore, if Q is a Π_1^0 class, then S may be taken to be a c. e. set.*

Proof. For Q in $2^{\mathbb{N}}$ define S to be the complement of T_Q. For the $2^{\mathbb{Z}}$ case, let $S = \bigcup_{n \in \mathbb{N}} \{0,1\}^{2n+1} \setminus T_Q$. □

An element X of a set P in either $2^{\mathbb{N}}$ or in $2^{\mathbb{Z}}$ is said to be *isolated* in P if there is a clopen set U such that $P \cap U = \{X\}$, equivalently, if there exists $i \leq j$ such that $P \cap \{Y : Y[i,j] = X[i,j]\} = \{X\}$. For any compact $P \subseteq \mathbb{N}^{\mathbb{N}}$, define the *the Cantor-Bendixson derivative* $D(P)$, to be the set of non-isolated elements of P. This derivative can be applied iteratively to define $D^{\alpha}(P)$ for any ordinal α.

1. $D^0(P) = P$
2. $D^{\alpha+1}(P) = D(D^{\alpha}(P))$
3. $D^{\lambda}(P) = \bigcap_{\alpha<\lambda} D^{\alpha}(P)$ for limit ordinals λ.

For any ordinal α and any compact P, $D^{\alpha}(P)$ is also compact. The classical *Cantor-Bendixson (CB) rank* of a countable compact set P is defined to be the least ordinal α such that $D^{\alpha+1}(P) = D^{\alpha}(P)$; $D^{rk(P)}(P)$ is the *perfect kernel* $K(P)$ of P. For a countable set P, the kernel of P is the empty set, so that $rk(P)$ is the least ordinal α such that $D^{\alpha}(P) = \emptyset$. Thus for any closed set P, P has rank zero if and only if P is empty and P has rank one if and only if P is finite. The Cantor-Bendixson (CB) rank $rk_P(X)$ of an element X in any class P is defined, for $X \notin K(P)$, as the least ordinal α such that $X \notin D^{\alpha+1}(P)$. Thus in particular $rk_P(X) = 0$ if and only if X is isolated in P. Under these definitions, $rk(P)$ is the supremum of $\{rk_P(X) + 1 : X \in P\}$ for countable P. Note that if α is a limit ordinal, then $rk_P(X)$ cannot equal α and that if P is compact, then $rk(P)$ cannot equal α.

We note that in several previous articles on effectively closed sets [5, 8, 7, 10] (including the conference version of the present paper), a different definition is given for the CB rank of a countable closed set, namely the the least ordinal α such that $D^{\alpha+1}(P) = \emptyset$. This will always be one less than $rk(P)$ as defined above. This alternative definition allows the $rk(P)$ for $P \subseteq 2^{\mathbb{N}}$ to be any countable ordinal and makes $rk(P)$ the supremum of $\{rk_P(X) : X \in P\}$. Since some of the results of the present paper apply to subshifts which are not necessarily effectively closed, we will use the classical definition.

For more background on computability and on the Cantor-Bendixson derivative, see [10], which includes the following (Lemma 4.2).

Lemma 2.3. *Let F be a continous map from $2^{\mathbb{N}}$ into $2^{\mathbb{N}}$ and let P and Q be closed sets such that $F[P] = Q$. Then for any $Y \in Q$, $rk_Q(Y) \leq \max\{rk_P(X) : X \in P \ \&\ F(X) = Y\}$.*

Note that the lemma also holds for the continuous functions from $2^{\mathbb{Z}}$ into $2^{\mathbb{Z}}$ since $2^{\mathbb{Z}}$ is homeomorphic to $2^{\mathbb{N}}$.

We will also need the following (essentially Theorem 4.1 of [4]). Let $\alpha \oplus \beta$ denote the *Hessenberg sum* of ordinals α and β. For our purposes, it suffices to note that for natural numbers m and n, $m \oplus n = m + n$ for finite ordinals and $\omega \oplus n = n \oplus \omega = \omega + n$.

Lemma 2.4. *For any closed sets P and Q in $2^{\mathbb{N}}$, any $X \in P$ and any $Y \in Q$, the Cantor-Bendixson rank of $X^{-}.Y$ in $P \otimes Q$ equals $rk_P(X) \oplus rk_Q(Y)$. Hence if $rk(P)$ and $rk(Q)$ are finite, $rk(P \otimes Q) = rk(P) + rk(Q) - 1$.*

An element X of $2^{\mathbb{N}}$ is said to be *periodic* if $X = v^{\omega}$ for some finite string v; the period of X is the minimal length of v such that $X = v^{\omega}$. X is said to be *eventually periodic* if for some strings u and v, $X = u^\frown v^{\omega}$ An element Z of $2^{\mathbb{Z}}$ is *periodic* if $Z = v^{\infty}$ for some finite v; the period of Z is the minimal $|v|$ such that $Z = v^{\infty}$. Z is *eventually periodic* if for some finite u, v and w, $Z = w^{-\omega}.u^\frown v^{\omega}$.

The following facts about periodic sequences will be useful.

Lemma 2.5. *Let u and v be finite words and let $X = v^{\omega}$ where $|v|$ is minimal. If $X = u^\frown v^{\omega}$, then $u = v^m$ for some m.*

Proof. Suppose that $X = u^\frown v^{\omega} = v^{\omega}$. If $u = \lambda$, then $u = v^0$. Otherwise, $u^{\omega} = v^{\omega}$ and therefore $|u| \geq |v|$ by the minimality of $|v|$ and it follows that $u = v^m$ for some m. $\square$

We will need the following simple connection between periodicity and the shift.

Lemma 2.6. *(a) $X \in 2^{\mathbb{N}}$ is periodic if and only if $\sigma^n(X) = X$ for some n.*

(b) $X \in 2^{\mathbb{N}}$ is eventually periodic if and only if $\sigma^{m+n}(X) = \sigma^m(X)$ for some m and n.

(c) $Z \in 2^{\mathbb{Z}}$ is periodic if and only if $\sigma^n(Z) = Z$ for some n.

Proof. We will just give the proof of (c). Suppose first that Z is periodic and let $Z = u^{\infty}$ for some finite string u of length n. Then we have $Z[kn,(k+1)n) = u$ for all $k \in \mathbb{Z}$. So, $\sigma^n(Z[kn,(k+1)n)]) = Z[(k+1)n,(k+2)n)] = u$ for all $k \in \mathbb{Z}$. Thus, $\sigma^n(Z) = u^{\infty} = Z$ $\square$

3 Countable Subshifts

This section contains results on the computability and decidability of subshifts of rank two. We examine the connection between the shift operator and the Cantor-Bendixson derivative. This leads to the surprising result that there are no subshifts of Cantor-Bendixson rank exactly $\omega + 1$. The following lemmas will be needed.

Lemma 3.1. *(a) If $Q \subseteq 2^{\mathbb{N}}$ is a finite subshift, then Q contains a periodic element and every element of Q is eventually periodic.*

(b) If $Q \subseteq 2^{\mathbb{Z}}$ is a finite subshift, then every element of Q is periodic.

Proof. (a) Let $Y \in Q$, where Q is a finite subshift. Then for each i, $\sigma^i(Y) \in Q$. Since Q is finite, there must exist m and n such that $\sigma^{m+n}(Y) = \sigma^n(Y)$, so that Y is eventually periodic. Then $X = \sigma^m(Y)$ is a periodic member of Q.

For (b), let $Z \in Q$. Since Q is finite, there exist $m < n$ such that $\sigma^m(Z) = \sigma^n(Z)$. Thus, $Z = \sigma^{-m}(\sigma^n(Z)) = \sigma^{n-m}(Z)$. Hence Z is periodic. $\square$

Example 3.2. *The results are different for $2^{\mathbb{N}}$ and $2^{\mathbb{Z}}$ since part (b) does not hold in $2^{\mathbb{N}}$, as seen by the example of $\{0^\omega, 10^\omega\}$.*

Lemma 3.3. *Let $P \subseteq 2^{\mathbb{N}}$ (or $P \subseteq 2^{\mathbb{Z}}$) be any closed set. Then $D^\alpha\sigma(P) = \sigma(D^\alpha(P))$ for any ordinal α. Hence, if $P \subseteq 2^{\mathbb{N}}$ is a subshift, then, for all $X \in P$, $rk_P(X) \leq rk_P(\sigma(X))$. Furthermore, if $P \subseteq 2^{\mathbb{Z}}$ is a subshift, then, for all $X \in P$, $rk_P(X) = rk_P(\sigma(X))$.*

Proof. The key to the proof is the case when $\alpha = 1$. Let $P \subseteq 2^{\mathbb{N}}$ be a subshift.

Suppose first that $X \in D(\sigma(P))$. Then there is a sequence $\{Y_n\}_{n\in\mathbb{N}} \subseteq P$ of distinct members of P such that $\lim_n \sigma(Y_n) = X$. For each n, either $Y_n(0) = 0$ or $Y_n(0) = 1$. Thus there exists $i \in \{0,1\}$ and an infinite subsequence $n_0, n_1, \ldots$ such that $Y_{n_k}(0) = i$ for all k. It follows that $\lim_k Y_{n_k} = i^\frown X$ and belongs to $D(P)$. But then we have $\sigma(i^\frown X) = X$ and hence $X \in \sigma(D(P))$.

Conversely, suppose that $X \in \sigma(D(P))$ and choose $Y \in D(P)$ such that $\sigma(Y) = X$. Then there is a sequence $\{Y_n\}_{n\in\mathbb{N}} \subseteq P$ of distinct members of P such that $\lim_n Y_n = Y$. It follows from the continuity of the shift operator that $\sigma \lim_n \sigma(Y_n) = \sigma(Y) = X$ and hence $X \in D(\sigma(P))$.

For the space $2^{\mathbb{Z}}$, the converse argument is the same, but the first direction is simpler, since $\lim_n \sigma(Y_n) = X$ implies that $\lim_n Y_n = \sigma^{-1}(X)$, so that $\sigma^{-1}(X) \in D(P)$ and hence $X \in \sigma(D(P))$.

The proof proceeds by induction on α. For the successor case, we have

$$D^{\alpha+1}(\sigma(P)) = D(D^\alpha(\sigma(P))) = D(\sigma(D^\alpha(P))) = \sigma(D(D^\alpha(P))) = \sigma(D^{\alpha+1}(P)).$$

For the limit case, we have

$$D^\alpha(\sigma(P)) = \bigcap_{\beta<\alpha} D^\beta(\sigma(P)) = \bigcap_{\beta<\alpha} \sigma(D^\beta(P)) = \sigma(\bigcap_{\beta<\alpha} D^\beta(P)) = \sigma(D^\alpha(P)).$$

For the third equality, observe that if $Y = \sigma(X_\beta)$ with $X_\beta \in D^\beta(P)$ for each $\beta < \alpha$, then in $2^{\mathbb{N}}$, there exists $i < 2$ and a set $B \subseteq \alpha$ cofinal in α such that $X_\beta(0) = i$ for all $\beta \in B$ and hence $i^\frown Y \in D^\beta(P)$ for all $\beta \in B$. Thus $i^\frown Y \in \bigcap_{\beta<\alpha} D^\beta(P)$ (since this is a decreasing intersection) and hence $Y \in \sigma(\bigcap_{\beta<\alpha} D^\beta(P))$.

For the final conclusion, suppose that $rk_P(X) = \alpha$. Then $X \in D^\alpha(P)$ and hence $\sigma(X) \in \sigma(D^\alpha(P)) = D^\alpha(\sigma(P)) \subseteq D^\alpha(P)$. For $P \subseteq 2^{\mathbb{Z}}$, we note that the proof above can be modified to show that $D^\alpha(\sigma^{-1}(P)) = \sigma^{-1}(D^\alpha(P))$ and $rk_P(X) \leq rk_P(\sigma^{-1}(X))$. $\square$

Proposition 3.4. *For any subshift $Q \subseteq 2^{\mathbb{N}}$ (or $Q \subseteq 2^{\mathbb{Z}}$) and any ordinal α, $D^\alpha(Q)$ is a subshift.*

Proof. Let $Q \subseteq 2^{\mathbb{N}}$ ($2^{\mathbb{Z}}$). For any $X \in D^{\alpha}(Q)$, it follows from Lemma 3.3 that $\sigma(X) \in D^{\alpha}(\sigma(Q))$ and hence $\sigma(X) \in D^{\alpha}(Q)$. ☐

We will need the following lemmas relating subshifts of $2^{\mathbb{Z}}$ to the corresponding subshifts of $2^{\mathbb{N}}$. Let π_0 and π_1 be the two projection maps from $2^{\mathbb{Z}}$ onto $2^{\mathbb{N}}$ so that if $Z = X^{-}.Y$, then $\pi_1(Z) = Y$ and $\pi_0(Z) = X$; that is, for all n, $Y(n) = Z(n)$ and $X(n) = Z(-n-1)$.

Lemma 3.5. *Let $Q \subseteq 2^{\mathbb{Z}}$ be a subshift.*

(a) Let S be the set of finite strings u such that no element of Q has u as factor and let $S^{-} = \{u^{-} : u \in S\}$. Then $Q = Q^{\mathbb{Z}}_{S}$, $\pi_1[Q] = Q^{\mathbb{N}}_{S}$ and $\pi_0[Q] = Q^{\mathbb{N}}_{S^{-}}$.

(b) $\pi_0[Q]$ and $\pi_1[Q]$ are subshifts of $2^{\mathbb{N}}$ and are effectively closed if Q is effectively closed.

(c) $Q \subseteq \pi_0[Q] \otimes \pi_1[Q]$.

Proof. Let S be as defined. It is clear that $Q \subseteq Q^{\mathbb{Z}}_{S}$. On the other hand, we know by Proposition 2.2 that $Q = Q^{\mathbb{Z}}_{R}$ for some R and we must have $R \subseteq S$, so that $Q^{\mathbb{Z}}_{S} \subseteq Q^{\mathbb{Z}}_{R} = Q$.

Observe next that $\pi_1[Q]$ is a closed set as the continuous image of the closed set Q and is effectively closed if Q is effectively closed, since π_1 is a computable mapping. Furthermore, $\pi_1[Q]$ is a subshift. That is, if $Y = \pi_1(Z)$, then $\sigma(Y) = \pi_1(\sigma(Z))$. Thus by Proposition 2.2, we have $\pi_1[Q] = Q^{\mathbb{N}}_{R}$ for some R and we may assume that $S \subseteq R$, since certainly $\pi_1[Q] \subseteq Q^{\mathbb{N}}_{S}$. On the other hand, if $u \notin S$ then for some $Z \in Q$, u is a factor of Z and by shifting Z if necessary we can obtain u as a factor of $\pi_1(Z)$, so that $u \notin R$. It follows that $\pi_1[Q] = Q^{\mathbb{N}}_{S}$. A similar argument holds for $\pi_0[Q]$.

For part (c), let $Z = X^{-}.Y \in Q$. Then Z avoids S, so that X avoids S^{-} and Y avoids S. ☐

Proposition 3.6. *(a) For any subshift $Q \subseteq 2^{\mathbb{N}}$ of rank $\alpha + 1$, $D^{\alpha}(Q)$ has a periodic element and any element of Q having rank α is eventually periodic.*

(b) If $Q \subseteq 2^{\mathbb{Z}}$ has rank $\alpha + 1$, then all elements of rank α are periodic.

Proof. (a) Let Q have rank $\alpha + 1$, so that $D^{\alpha+1}(Q) = \emptyset$ and $D^{\alpha}(Q)$ is finite. Then $D^{\alpha}(Q)$ is a subshift by Proposition 3.4 and the result follows by Lemma 3.1 (a).

(b) The same argument works as in (a).

☐

Next we consider some results on the decidability of subshifts.

Lemma 3.7. *For any $Z \in 2^{\mathbb{N}}$ (or in $2^{\mathbb{Z}}$) which is eventually periodic, the set of factors of Z is decidable.*

Proof. We will give the proof for $2^{\mathbb{N}}$ and leave the slightly more complicated proof for $2^{\mathbb{Z}}$ to the reader. Suppose that $X = vw^{\omega}$ and let W be the set of factors of X. Then $x \in W$ if and only if it has one of the following forms:

(i) $v[s,t)$ where $0 \leq s \leq t \leq |v|$

(ii) $v[s,|v|)w^n w[0,t)$ where $s < |v|$, $n \geq 0$ and $t \leq |w|$

(iii) $w[s,|w|)w^n w[0,t]$ where $s < |w|$, $t < |w|$, and $n \geq 0$.

The possible choices of n here are bounded by $|x|$, so that there is a finite algorithm for checking whether $x \in W$. This shows that W is decidable, as required. $\square$

Note that if X is computable, then in general the set of factors of X is *not* decidable.

Example 3.8. *Let $E \subseteq \mathbb{N}$ be any set which is c. e. but not computable and let $E = \{n_0, n_1, \dots\}$ be a computable enumeration without repetition. Let $X = 10^{n_0}10^{n_1}1\dots$. Then X is computable but the set of factors of X is not since 10^n1 is a factor of X if and only if $n \in E$.*

Proposition 3.9. *Given any natural number m, there is an at most countable decidable subshift $P \subseteq 2^{\mathbb{N}}$ $(2^{\mathbb{Z}})$ such that its rank is equal to m and all of its elements are eventually periodic.*

Proof. Let $P_0 = \emptyset$ and for each n, let P_{n+1} be the set of elements of $2^{\mathbb{N}}$ containing at most n ones. Then each P_{n+1} is clearly a subshift and is decidable since $v \in T_{P_{n+1}}$ if and only if v contains at most n ones.

Certainly $P_0 = \emptyset$ has rank 0 and $P_1 = \{0^{\omega}\}$ has rank one. For each n, we claim that $P_n \subset P_{n+1}$ and $D(P_{n+1}) = P_n$.

Suppose first that $X \in P_n$ and let $X = v^\frown 0^{\omega}$ where v has at most $n-1$ ones. Then $X = lim_i X_i$ where $X_i = v^\frown 0^i 1^\frown 0^{\omega} \in P_{n+1}$. Hence $X \in D(P_{n+1})$. This shows that $D(P_{n+1}) = P_n$ and it follows by induction that P_{n+1} has rank n.

Next suppose that $X \notin P_n$. If $X \notin P_{n+1}$, then certainly $X \notin D(P_{n+1})$, so we may assume that $X \in P_{n+1}$. Then X must have exactly n ones, so that $X = v^\frown 0^{\infty}$ where v has exactly n ones. Then clearly $P_{n+1} \cap [v] = \{X\}$, so that $X \notin D(P_{n+1})$.

The same construction also works for $2^{\mathbb{Z}}$. $\square$

Note that for the sequence of sets P_n defined above, $\bigcup_n P_n$ is not closed and in fact is dense in $2^{\mathbb{N}}$.

The following lemma is well-known in the area of combinatorics on words. For example, it follows easily from Theorem 1.3.13 of [15, p. 22]. A proof for $2^{\mathbb{N}}$ is given in [7].

Lemma 3.10. *Suppose $X \in 2^{\mathbb{N}}$ or $X \in 2^{\mathbb{Z}}$ is not eventually periodic. Then for any $k \in \mathbb{N}$, there are at least $k+1$ distinct factors of length k that occur infinitely often in X.*

Proof. We only indicate how to prove the result for $2^{\mathbb{Z}}$ from the corresponding result for $2^{\mathbb{N}}$. Let $Z \in 2^{\mathbb{Z}}$ and let $Z = X^{-}.Y$ where X and Y are in $2^{\mathbb{N}}$. If Z is not eventually periodic, then at least one of the two elements X and Y of $2^{\mathbb{N}}$ are not eventually periodic. The result now follows from the $2^{\mathbb{N}}$ case above. □

Theorem 3.11. *For any subshift $Q \subseteq 2^{\mathbb{N}}$ ($2^{\mathbb{Z}}$) of rank two, every member of Q is eventually periodic.*

Proof. Suppose $Q \subseteq 2^{\mathbb{N}}$ is a subshift, and suppose, by way of contradiction, that $X \in Q$ and that X is not eventually periodic. Let k be arbitrary and let $w_0, \dots, w_k$ be distinct factors of X of length k which occur infinitely often in X. Then for each $i \leq k$, there are infinitely many n such that $\sigma^n(X) \in [w_i] \cap Q$. Since X is not eventually periodic, $m \neq n$ implies that $\sigma^m(X) \neq \sigma^n(X)$. Thus $[w_i] \cap Q$ has a limit point for each $i \leq k$. It follows that Q has at least $k+1$ limit points. Since k was arbitrary, Q has infinitely many limit points and thus $rk(Q) > 2$, a contradiction.

For the $2^{\mathbb{Z}}$ case, the argument is similar. Again, for a given k, we have $k+1$-many distinct factors of Z of length k which occur infinitely often in Z, say $w_0, w_1, ..., w_k$. For each $i \leq k$, this implies that there are an infinite number of distinct members of Q which have w_i as a central block. Then by compactness Q must have a limit point having w_i as a central block. Since each w_i for $0 \leq i \leq k$ is distinct, we obtain $k+1$ distinct limit points. Since k was arbitrary, Q has infinitely many limit points and thus $rk(Q) > 2$. □

We note here that there are Π^0_1 classes of rank two with noncomputable elements [10]. (Note that due to a different defintion of rank in [10], these classes are said there to have rank one). Hence we have the following.

Corollary 3.12. *There is a Π^0_1 class of rank two which is not degree-isomorphic to any subshift of rank two.*

Next we will discuss the decidability of rank two subshifts.

Theorem 3.13. *(a) For any Turing degree* $\mathbf{d}$*, there is a subshift $Q \subseteq 2^{\mathbb{N}}$ of rank two such that T_Q has degree* $\mathbf{d}$.

(b) For any c. e. degree $\mathbf{d}$*, there is a Π^0_1 subshift $Q \subseteq 2^{\mathbb{N}}$ of rank two such that T_Q has degree* $\mathbf{d}$.

Proof. Let A be any set of natural numbers of degree $\mathbf{d}$ and let Q contain limit points 0^ω and $1^\frown 0^\omega$, along with isolated points $0^n 1^\frown 0^\omega$, for $n > 0$ and $1^\frown 0^n 1^\frown 0^\omega$ for $n \in A$. Then Q is a rank two subshift and we have $1^\frown 0^n 1 \in T_Q$ if and only if $n \in A$. Thus $A \leq_T T_Q$. For the other direction, just observe that no string with more than two 1's belongs to T_Q and every string with one or no 1's belongs to T_Q.

For (b), just take a c. e. set B of degree $\mathbf{d}$, let $A = \mathbb{N} - B$ and construct Q as in (a). □

Our next theorem will show that we cannot achieve the result of the Theorem 3.13 for subshifts of $2^{\mathbb{Z}}$. However, in the next section we will present a similar result for rank three subshifts of $2^{\mathbb{Z}}$.

Theorem 3.14. *Let $Q \subseteq 2^{\mathbb{Z}}$ be a subshift of rank two. Then Q can be decomposed as a finite number of periodic elements together with a finite number of elements of type $u^{-\omega}.v^\frown w^\omega$ and their orbits under the shift map.*

Proof. Since Q is of rank two, $D(Q)$ has finitely many points and they are periodic by Lemma 3.1. Next, consider the isolated points. By Theorem 3.11 they are eventually periodic. Thus every element of Q is eventually periodic and hence is computable.

It should be noted that although every limit point is periodic, there may also be periodic points which are isolated. Nevertheless we will argue that Q has only finitely many periodic points. Assume for a contradiction that the set A of periodic points of Q is infinite. Then, by compactness, there exist a limit point of A, say X. Moreover, X must be periodic, since it belongs to the finite shift $D(Q)$; let $X = v^\infty$.

Our goal is to construct a sequence $\{Z_k : k \in \omega\}$ of elements of A, which converges to a nonperiodic limit.

We begin with a $X = v^\infty = \lim_k u_k^\infty$, for a sequence $\{u_k : k \in \omega\}$ of distinct u_k, all different from v. We may assume without loss of generality that $|u_k| \geq k|v|$. Since $\lim_k u_k^\infty = v^\infty$, we may assume furthermore that $v^k \preceq u_k$.

Now for each k, let $n_k \geq k$ be the largest such that $v^{n_k} \preceq u_k$ and let $u_k = v^{n_k} w_k$ where $v \not\preceq w_k$. Now let $Z_k = \sigma^{n_k|v|}(u_k^\infty) \in A$ so that

$$Z_k = (w_k v^{n_k})^\infty = (w_k v^{n_k})^{-\omega}.w_k{}^\frown(v^{n_k} w_k)^\omega.$$

It is important to recall here that when we write, in general $Z = U^-.v^\frown W$, this means that $Z(i) = v(i)$ for $i < |v|$, that $Z(|v| + j) = W(j)$ for all j, and that $Z(-j-1) = U(j)$ for all j.

Since Z_k has period $\geq k$, it follows that $\{Z_k : k \in \omega\}$ is infinite and hence has a limit point Z. We may assume without loss of generality that the for each $i < j$, $Z_i \neq Z_j$. Since the sequence $\langle n_k \rangle_{k<\omega}$ tends to infinity, it follows that $Z = v^{-\omega}.Y$ for some Y. Since Z is a limit point of Q and hence must be periodic, it follows that $Y = v^\omega$.

We will show that this leads to a contradiction, in two cases.

Case I: Suppose that $|w_k| \geq |v|$ for infinitely many k. Since $\lim_k Z_k = v^\infty$, there is some K such that for $k \geq K$, Z_k begins with $v^{-\omega}.v$ and hence there is some k such that $v \preceq w_k$, a contradiction.

Case II: Suppose that $|w_k| < |v|$ for all but finitely many k. Then there is a fixed w with $|w| < |v|$ such that $Z_k = (wv^{n_k})^{-\omega}.w^\frown(v^{n_k}w)^\omega$ for infinitely many k. It follows that $v^\infty = \lim_k Z_k = v^{-\omega}.w^\frown v^\omega$. Then by Lemma 2.5, $w = \lambda$ and therefore infinitely many of the $Z_k = v^\infty$, again a contradiction.

Thus Q has in fact only a finite number of periodic points.

Next, we will analyze the set of nonperiodic (hence isolated) elements of Q. These are all of the form $u^{-\omega}.v^\frown w^\omega$. Note that for every element of Q of this form, both u^∞ and w^∞ are limit points of Q. Since $D(Q)$ is finite, there are only finitely many such u^∞ and w^∞. Now let S be the set of strings v such that there exist u and w with $u^{-\omega}.v^\frown w^\omega \in Q$ and such that u is not a prefix of v and w is not a suffix of v.

We claim that S is finite. Otherwise, by the above, there are fixed u and w such that $\{v \in S : u^{-\omega}.v^\frown w^\omega \in Q\}$ is infinite. We may assume that $|u|$ and $|w|$ is minimal here, that is, there is no proper prefix u_0 of u such that $u_0^\omega = u^\omega$ and similarly for w.

Then there will be an infinite sequence $\{(v_k) : k \in \mathbb{N}\}$ such that, for each k, u is not a prefix of v_k, w is not a suffix of v_k, $|v_k| < |v_{k+1}|$ and $Y_k = u^{-\omega}.v_k{}^\frown w^\omega \in Q$. It is easy to see that if $j \neq k$, then $Y_j \neq Y_k$.

That is, suppose that $j < k$ but $Y_j = Y_k$. Then $v_j{}^\frown w^\omega = v_k{}^\frown w^\omega$. Now by deleting the first $|v_j|$ terms of v_k, we obtain $w^\omega = v^\frown w^\omega$, where v is a nonempty suffix of v_k. It now follows from Lemma 2.5 that $v = w^m$ for some $m > 0$. But this implies that w is a suffix of v_k, which is a contradiction.

It now follows that this infinite set $\{Y_i : i \in \mathbb{N}\}$ has a limit point $Z \in Q$ of the form $u^{-\omega}.X$ with $u \not\prec X$. But this means that Z is a nonperiodic limit point of Q, a contradiction. Hence S must be finite.

It follows that Q consists of finitely many periodic points together with the shifts of a finite set $C = \{X_i = u_i^{-\omega}.v_i{}^\frown w_i^\omega : i < n\}$. ☐

Corollary 3.15. *Let $Q \subseteq 2^{\mathbb{Z}}$ be a subshift of rank two. Then, Q is decidable and every element of Q is computable.*

Proof. It follows from Theorem 3.14 that Q consists of finitely many periodic points together with the shifts of a finite set $C = \{X_i = u_i^{-\omega}.v_i{}^\frown w_i^\omega : i < n\}$. Then a finite string x belongs to the bi-tree T_Q if and only if it is a factor of one of the elements of C. This now implies that T_Q is decidable. That is, for each i, the set of factors of X_i is decidable by Lemma 3.7 and there are only finitely many X_i, so that T_Q is a finite union of decidable sets. ☐

We will next consider a special case in which rank two subshifts of $2^{\mathbb{N}}$ are decidable. The following lemma is needed.

Lemma 3.16. *Let $Q \subseteq 2^{\mathbb{N}}$ be a subshift, let $X = v^\omega$ be a periodic element of Q with period k and, for each $i < k$ and each n, let $Q_{i,n} = \{Z : v^{n\frown}(v \upharpoonright i)^\frown(1 - v(i))^\frown Z \in Q\}$. Then $Q_{i,n+1} \subseteq Q_{i,n}$ for each n.*

Proof. If $Z \in Q_{i,n+1}$, then $v^{n+1}(v \upharpoonright i)(1 - v(i))^\frown Z \in Q$, so that, since Q is a subshift, $v^n(v \upharpoonright i)(1 - v(i))^\frown Z \in Q$ and therefore $Z \in Q_{i,n}$. ☐

Theorem 3.17. *Let $Q \subseteq 2^{\mathbb{N}}$ be a subshift of rank two such that every element of $D(Q)$ is periodic. Then Q is decidable and every element of Q is computable.*

Proof. Let $X = v^\omega$ be a periodic element of $D(Q)$ with period k. Let $Q_{i,n} \subseteq 2^{\mathbb{N}}$ be defined as in Lemma 3.16. Since X has rank one, there exists, for each $i < k$, some n such that $Q_{i,m}$ is finite for all $m \geq n$. To see this, suppose by way of contradiction that $Q_{i,n}$ is infinite for all n, let Z_n be a limit point in $Q_{i,n}$ and let $X_n = v^n\frown(v \upharpoonright i)\frown(1-v(i))\frown Z_n$ be the corresponding member of Q. Then each X_n is a limit point of Q and for $m < n$, $X_m \neq X_n$, since $X_m(m|v|+i) = 1-v(i)$ but $X_n(m|v|+i) = v(i)$. Observe that $X = \lim_n X_n$ which implies that X has rank greater than one, a contradiction.

Since $Q_{i,n+1} \subseteq Q_{i,n}$ for all i and n, it follows that the sequence $\{Q_{i,n} : n \in \mathbb{N}\}$ of finite sets is eventually constant and equal to some fixed finite subset P_i of $2^{\mathbb{N}}$. Let D_i be the decidable set T_{P_i}. Now let $S(v) = \{(i,n) : i < k \ \&\ Q_{i,n} \text{ is finite}\}$ and let $A(v)$ be the set of strings of the form $v^n(v \upharpoonright i)\frown w$ for some $(i,n) \in S(v)$ and some w. Then $S(v)$ is computable since it is a cofinite set. It follows that $A(v)$ is computable, since it is the union of finitely many computable sets together with $\{v^n(v \upharpoonright i)(1-v(i))\frown w : (i,n) \in S(v) \ \&\ w \in P_i\}$.

For each of the finitely many limit points $v_t^\omega \in Q$, we may similarly define the set $A(v_t)$ of strings in T_Q which branch off from v_t^ω where the appropriately defined set $Q_{i,n}$ is finite. We claim that T_Q is the union of the finitely many computable sets $A(v_t)$ together with the words of the form $v_t^n(v_t \upharpoonright i)$ for some i and n and is therefore decidable. Certainly each such string is in T_Q. Now suppose that u is some string in T_Q which is not an initial segment of any of the limit points. Choose v_t so that u has the longest agreement with v_t^ω of the limit points and choose i and n so that $v_t^n\frown(v_t \upharpoonright i)(1-v_t(i)) \preceq u$. Then $v_t^n(v_t \upharpoonright i)(1-v_t(i))$ disagrees with every limit point so that $Q_{i,n}$ is finite and hence $u \in A(v_t)$. □

Corollary 3.18. *For any subshift $Q \subseteq 2^{\mathbb{N}}$ of rank two, there is some finite n such that $\sigma^n(Q)$ is decidable.*

Proof. By Theorem 3.11, $D(Q)$ is a finite set of eventually periodic points. For each $X \in D(Q)$, $\sigma^m(X)$ is periodic for some m; just let n be the maximum m over $X \in D(Q)$. Then by Lemma 3.3, $D(\sigma^n(Q)) = \sigma^n(D(Q))$ and thus contains only periodic points, so that Theorem 3.17 applies. □

In the next section, we will show that a rank three subshift of $2^{\mathbb{N}}$ can have members which are not eventually periodic and indeed not even computable.

There is another interesting consequence of Lemma 3.16. Recall that by compactness no subshift in $2^{\mathbb{N}}$ can have rank ω.

Theorem 3.19. *There is no subshift of rank $\omega+1$ in $2^{\mathbb{N}}$ (or in $2^{\mathbb{Z}}$).*

Proof. Let $Q \subseteq 2^{\mathbb{N}}$ be a subshift and suppose by way of contradiction that Q has rank $\omega+1$. Then $D^{\omega+1}(Q) = \emptyset$ and $D^\omega(Q)$ is finite. Then there is a periodic element X of rank ω by Proposition 3.6. Let X have period k and let the sets $Q_{i,n}$ be defined as in Lemma 3.16. Since X has rank ω, there is some n such that for all i and all $m \geq n$, $Q_{i,m}$ has rank $< \omega$. To see this, suppose by way of contradiction that $Q_{i,m}$ has rank $\geq \omega$ (and hence rank $= \omega+1$) for all m and let Z_m be an element of $Q_{i,m}$ with rank $\geq \omega$. Then as in the proof of Theorem 3.17, let $X_m = v^m\frown(v \upharpoonright i)(1-v(i))\frown Z_m$ and observe that each X_m has rank $\geq \omega$ and that $X = \lim_m X_m$ therefore has rank of rank $\geq \omega+1$, a contradiction.

Now suppose that $Q_{i,n}$ has rank $r_i < \omega$ and let $r = \max\{r_i : i \leq k\}$. Then by Lemma 3.16, $rk(Q_{i,m}) \leq r$ for all $m > n$. But this implies that $rk(X) \leq r + 1$, which is the desired contradiction.

Next assume by way of contradiction that $Q \subseteq 2^{\mathbb{Z}}$ is a subshift of rank $\omega + 1$. Let S be the set of finite strings u such that no element of Q has u as factor, and let $Q_0 = Q^{\mathbb{N}}_{S^-}$ and $Q_1 = Q^{\mathbb{N}}_S$. Then by Lemma 3.5, $Q = Q^{\mathbb{Z}}_S$, $\pi_0[Q] = Q_0$, and $\pi_1[Q] = Q_1$. It follows from [the note after] Lemma 2.3 that Q_0 and Q_1 have rank $\leq \omega$. Then by the previous paragraph, Q_0 and Q_1 must have finite rank, so that, by Lemma 2.4, $Q_0 \otimes Q_1$ has finite rank. But $Q \subseteq Q_0 \otimes Q_1$, and hence Q must also have finite rank. $\square$

It follows from the proof that there is no subshift of rank $\alpha + 1$ in $2^{\mathbb{N}}$ or $2^{\mathbb{Z}}$, for any limit ordinal α.

Example 3.20. *There is a Π^0_1 subshift $Q \subseteq 2^{\mathbb{Z}}$ of rank $\omega + 2$. For any $Z \in 2^{\mathbb{Z}}$, let $m(Z)$ be the minimum such that 10^m1 is a factor of Z and let $n(Z)$ be the number of 1's in Z. Now define Q so that $Z \in Q$ iff $n(Z) \leq m(Z)$ or if $n(Z) \leq 1$. Thus every member of Q has only finitely many 1's. It is easy to see that, for any k, $D^k(Q) = \{Z : n(Z) \leq m(Z) - k\}$. This means that $Z \in Q$ is isolated if $n(Z) = m(Z)$ and in general has rank k if $n(Z) = m(Z) - k$. It follows that $D^{\omega}(Q) = \{Z : n(Z) \leq 1\}$ and that $D^{\omega+1}(Q) = \{0^{\infty}\}$. This same example also works in $2^{\mathbb{N}}$.*

4 Subshifts of Rank Three and Four

In this section, we examine the complexity of subshifts of rank three and four and the complexity of their elements. We also give a general result showing that for any closed set $P \subseteq 2^{\mathbb{N}}$ of arbitrary rank $\alpha + 1$, there is a subshift Q of rank $\alpha + 3$ and a computable injection from P into Q. We begin with the counterpart of the Theorem 3.13 for subshifts of $2^{\mathbb{Z}}$.

Theorem 4.1. *(a) For any Turing degree $\mathbf{d}$, there is a subshift $Q \subseteq 2^{\mathbb{Z}}$ of rank three such that T_Q has degree $\mathbf{d}$*

(b) For any c. e. degree $\mathbf{d}$, there is a Π^0_1 subshift $Q \subseteq 2^{\mathbb{Z}}$ of rank three such that T_Q has degree $\mathbf{d}$.

Proof. (a) Let $A \subseteq \mathbb{N}$ be an infinite set of degree $\mathbf{d}$. $Q \subseteq 2^{\mathbb{Z}}$ is defined as follows. For any $w \in \{0,1\}^*$, $w \in T_Q$ if and only if

(i) w has at most two 1's.

(ii) For $i > j$, if $w(i) = w(j) = 1$, then $i - j \in A$.

Then Q has the following elements.

(0) For any $i \in \mathbb{Z}$ and $n \in A$, Q has the isolated element $X_{i,n}$ with $X_{i,n}(j) = 1 \iff j = i \vee j = i + n$.

(1) For any i, Q has the element X_i of rank one, where $X_i(j) = 1 \iff j = i$.

(2) Q has a unique element of rank two, namely 0^∞.

(b) If $\mathbf{d}$ is a c. e. degree, let A be a co-c. e. set of degree $\mathbf{d}$ in the argument above. Then T_Q is a Π^0_1 set and hence Q is a Π^0_1 class. □

We observe that for the subshift Q constructed in Theorem 4.1 corresponding to the set A, Q will be decidable if and only if the set A is computable, since T_Q has the same Turing degree as A.

Proposition 4.2. *For any increasing sequence $n_0 < n_1 < \dots$,*

(a) There is a subshift $Q \subseteq 2^{\mathbb{N}}$ of rank three which has a unique element, $0^{n_0}1^\frown 0^{n_1}1\dots$, of rank two.

(b) There is a subshift $Q \subseteq 2^{\mathbb{Z}}$ of rank three which has a unique element, $0^{-\omega}.1^\frown 0^{n_0}1^\frown 0^{n_1}1\dots$, of rank two.

Proof. (a) The subshift $Q \subseteq 2^{\mathbb{N}}$ will have the following elements:

(0) For each $k \geq 0$ and each $n \leq n_k$, the isolated element $0^n1^\frown 0^{n_{k+1}}1^\frown 0^{n_{k+2}}1\dots$

(1) For every n, the element $0^n1^\frown 0^\omega$ which will have rank one in Q.

(2) 0^ω, which is the unique element of rank 2 in Q.

Note that the elements of type (0) are all iterated shifts of $X = 0^{n_0}1^\frown 0^{n_1}1\dots$.

Here we see that Q avoids the set S consisting of

(i) all words 10^n1 such that $n \notin \{n_1, n_2, \dots\}$;

(ii) all words $0^n1^\frown 0^{n_i}1$ such that $n \geq n_i$;

(iii) all words $10^{n_i}1^\frown 0^{n_j}1$ such that $j \neq i+1$;

(iv) all words $10^{n_i}1^\frown 0^{n_{i+1}+1}$.

Items (iii) and (iv) ensure that each block $10^{n_i}1$ must be followed by the block $0^{n_{i+1}}1$ and can only be preceded by the block $10^{n_{i-1}}$ (if $i > 1$).

(b) The subshift $Q \subseteq 2^{\mathbb{Z}}$ will have the following elements:

(0) $\sigma^z(X)$ for each $z \in \mathbb{Z}$ is an isolated element

(1) $\{Y_i : Y_i(k) = 1 \iff k = i, i \in \mathbb{Z}\}$ is the set of rank one elements.

(2) 0^∞ is the unique rank 2 element of Q.

Here we see that Q avoids the set S as above except for $1^\frown 0^{n_0}1$. □

Theorem 4.3. *For any Turing degree $\mathbf{d}$, there is a rank three subshift $Q \subseteq 2^{\mathbb{N}}$ ($2^{\mathbb{Z}}$) which contains a member of Turing degree $\mathbf{d}$ and such that T_Q has Turing degree $\mathbf{d}$.*

Proof. Let $A = \{a_0, a_1, \dots\}$ be any infinite set of degree $\mathbf{d}$ and let $n_i = a_0 + a_1 + \dots + a_i$ for each i. Now apply Proposition 4.2. $\square$

This result can be improved as follows.

Theorem 4.4. *For any countable set $\mathbf{D} = \{\mathbf{d}_i : i \in \mathbb{N}\}$ of Turing degrees (including $\mathbf{0}$), there is a rank three subshift $Q \subseteq 2^{\mathbb{N}}$ (or $\subseteq 2^{\mathbb{Z}}$) such that $Deg(Q) = \mathbf{D}$.*

Proof. Let $p_0 = 2, p_1 = 3, \dots$ enumerate the prime numbers in increasing order. For each i, choose a set $A_i = \{n_{i,0} < n_{i,1} < \dots\}$ of degree $\mathbf{d}_i$, let $m_{i,j} = p_i^{n_{i,j}+1}$ and let $X_i = 0^{m_{i,0}}10^{m_{i,1}}1\dots$. Then the closed subshift of $2^{\mathbb{N}}$ generated by the set $\{X_i : i \in \mathbb{N}\}$ will again contain a unique element 0^ω of rank two, will again contain elements 0^n10^ω of rank one, and will contain, for each i, each $k > 0$ and each $n \le m_{i,k}$,the isolated element $0^n1^\frown 0^{m_{i,k+1}}1^\frown 0^{m_{i,k+2}}1\dots$. The argument for $Q \subseteq 2^{\mathbb{Z}}$ follows as in the proof of Proposition 4.2. $\square$

For effectively closed subshifts, the result is quite different.

Theorem 4.5. *If $Q \subseteq 2^{\mathbb{N}}$ is a Π^0_1 subshift of rank three, then all of its members are computable.*

Proof. $D(Q)$ is a subshift of rank two and hence all of its members are eventually periodic and therefore computable. The remaining members of Q are isolated and therefore computable by Theorem 3.12 of [10]. $\square$

Results for Π^0_1 subshifts of rank four or higher with noncomputable elements can be obtained from the following general result.

Theorem 4.6. *Let α be a countable ordinal and let $P \subseteq 2^{\mathbb{N}}$ be any closed set (Π^0_1 class) of Cantor-Bendixson rank $\alpha + 1$. Then there exists a closed subshift (Π^0_1 class) $Q \subseteq 2^{\mathbb{N}}$ (and also $Q_1 \subseteq 2^{\mathbb{Z}}$) of rank $\alpha+3$, a computable injection from P into Q, and furthermore, a countable-to-one degree-preserving mapping from $Q - D^{\alpha+1}(Q)$ onto P. Furthermore, $D^{\alpha+1}(Q)$ is the set of eventually periodic points of Q.*

Proof. We will uniformize the method of Proposition 4.2. For the first step, transform P into a closed set in which every element has the form $0^{n_0}1^\frown 0^{n_1}1\dots$, by mapping an arbitrary $X \in 2^{\mathbb{N}}$ to $\Phi(X) = (0^{X(0)}1^\frown 0^{X(0)+X(1)+1}1\dots)$. Note that, for each i, $n_i + 1 \le n_{i+1} \le n_i + 2$ and hence $i \le n_i \le 2i + 1$. This map is a (truth-table) computable injection and therefore preserves computability and rank. Now define $Q \subseteq 2^{\mathbb{N}}$ to contain the following elements.

(0) For each $X \in P$ and each $i \in \mathbb{N}$, Q contains all of the shifts $\sigma^i(X)$. For any such $Y = \sigma^i(X)$, the rank of Y in Q will be

$$rk_Q(Y) = \max\{rk_P(X) : Y = \sigma^j(X) \text{ for some } j \in \mathbb{N} \text{ and some } X \in P\}$$

(1) For each n, Q contains the element $0^n1^\frown 0^\omega$, which will have rank $\alpha + 1$ in Q.

(2) Q contains the element 0^ω of rank $\alpha + 2$.

A crucial observation is that for any $Y = 0^{n_0}1^\frown 0^{n_1}1 \cdots \in Q$, $\{w : w^\frown Y \in P\}$ is finite. This is because if $w = 0^m$, then $m < n_1 - n_0$ and if $w = 0^{m_0}1 \ldots {}^\frown 0^{m_k}1^\frown 0^m$, then $m_0 < m_1 < \cdots < m_k < m + n_0 < n_1$.

It is clear that Q is closed under the shift operator, but we need to check that Q is topologically closed. Suppose therefore that Y is a limit of point of Q. If Y contains at most one 1, then $Y \in Q$, so without loss of generality Y has a prefix of type $0^{n_0}1^\frown 0^{n_1}1$. Now let $Y = \lim_i Y_i$ where each $Y_i \in Q$. It follows from the initial assumptions about P that Y has infinitely many 1's, say $Y = 0^{n_0}1^\frown 0^{n_1}1^\frown 0^{n_2}1 \ldots$ for some infinite sequence $\{n_i : i \in \mathbb{N}\}$; this is because any element of Q which extends $0^{n_0}1^\frown 0^{n_1}1$ can have at most $n_1 + 2$ zeroes before having a 1. Now we may assume that $0^{n_0}1 \ldots 0^{n_i}1 \prec Y_i$ for each i and that $Y_i = \sigma^{e_i}(X_i)$ for some $X_i \in P$ and $e_i \in \mathbb{N}$. That is, for each i, X_i begins with an initial segment of the form

$$0^{m_0}1^\frown 0^{m_1}1 \ldots 0^{m_k}1^\frown 0^{n_0}1 \ldots 0^{n_i}1.$$

Since $m_0 < m_1 < \cdots < m_k < n_0$ in every case, there are only finitely many possible segments $0^{m_0}1 \ldots 0^{m_k}1$ before Y_i begins. So, without loss of generality, we may assume that each X_i has the same fixed initial segment and hence there is also a fixed e such that $Y_i = \sigma^e(X_i)$ for all i. It follows that the sequence $\{X_i : i \in \mathbb{N}\}$ converges to a limit

$$X = \lim_i X_i = 0^{m_0}1 \ldots 0^{m_k}1^\frown Y$$

with $Y = \sigma^e(X)$. But P is closed and hence $X \in P$ so that $Y \in Q$.

We can now conclude from Proposition 2.2 that $Q = Q_S$, where $S = \{w : w \text{ is not a factor of any } Y \in Q\}$. If P is a Π^0_1 class, then we need to show that Q is also a Π^0_1 class. Let $R = \{w : w \text{ is not a factor of any } X \in P\}$. We claim that $S = R$. Since $P \subseteq Q$, it follows that $S \subseteq R$. Now suppose that $w \notin S$. Then w is a factor of some $Y \in Q$. If Y has at most one occurrence of 1, then w is a factor of every $X \in P$ by the initial assumption, so that $w \notin R$. Otherwise $Y = \sigma^e(X)$ for some $X \in P$, $e \in \mathbb{N}$ and hence w is a factor of X so that $w \notin R$.

If P is a Π^0_1 class, then T_P is a Π^0_1 set and we will show that R is a c. e. set. That is, given $w = 0^{m_0}1^\frown 0^{m_1}1 \ldots 0^{m_k}1^\frown 0^m$, we observe that if w is a factor of some $X = 0^{n_0}1 \cdots \in P$, then $m_k = n_i$ for some $i \leq m_k$ and therefore

$$|0^{n_0}1 \ldots 0^{n_i}1^\frown 0^m| \leq 2 + 4 + \cdots + 2m_k + 2 + m + 1 = m_k^2 + 3m_k + m + 3$$

Note that the function f, defined on finite strings by $f(w) = m_k^2 + 3m_k + m + 3$ whenever w is of the form $0^{n_0}1 \ldots 0^{n_i}1^\frown 0^m$ and $f(w) = |w|$ otherwise, is computable. Now, $w \in R$ if and only if for all v of length $\leq f(w)$, if w is a factor of v, then $v \notin T_P$. Hence R is a c. e. set, as desired.

It remains to examine the rank of the elements of Q. First note that since $P \subseteq Q$, $rk_P(X) \leq rk_Q(X)$ for all $X \in P$ and furthermore $rk_Q(\sigma^e(X)) \geq rk_Q(X)$ by Lemma 3.3, so that $rk_Q(\sigma^e(X)) \geq rk_P(X)$ for all $X \in P$, $e \in \mathbb{N}$. It follows that $rk_Q(0^n1^\frown 0^\omega) \geq \alpha + 1$ and hence $rk_Q(0^\omega) \geq \alpha + 2$.

For the other direction, we first observe that for any $Y = 0^{n_0}1^\frown 0^{n_1} \cdots \in Q$, $\{X \in P : \sigma^e(X) = Y$ for some $e\}$ is finite. This is because each such X has the form $w^\frown Y$ for some w and by the crucial observation above there are only finitely many possible such w. Thus $\{rk_P(X) : Y = \sigma^e(X)$ for some $e\}$ is finite and has a maximum. We now show by induction on $\beta = \max\{rk_P(X) : X \in P\ \&\ \sigma^e(X) = Y$ for some $e\}$ that $rk_Q(Y) = \beta$.

Suppose first that $\beta = 0$, that is whenever $Y = \sigma^e(X)$, then X is isolated in P. Suppose by way of contradiction that Y is *not* isolated in Q. Then once again we have $Y = \lim_i \sigma^{e_i}(X_i)$, for some $e_i \in \mathbb{N}$, so that as above (without loss of generality), there will be a fixed e and a limit $X = \lim_i X_i \in P$ such that $Y = \sigma^e(X)$. But X is isolated in P, which gives the desired contradiction.

Now suppose the claim holds for all ordinals $< \beta$ and let $Y = \sigma^e(X)$ where $e \in \mathbb{N}$ and X has the maximum rank β. Suppose by way of contradiction that Y has rank $\geq \beta + 1$ in Q. Then $Y = \lim_i Y_i$ where, for each i, $Y_i = \sigma^e(X_i)$ and $rk_Q(Y_i) \geq \beta$. It follows that $rk_P(X_i) \geq \beta$ since if $rk_P(X_i) < \beta$ then by the induction $rk(Y_i) < \beta$. It now follows as above (without loss of generality) that $X = \lim_i X_i$ which would imply that $rk_P(X) \geq \beta + 1$. This is the desired contradiction.

We may now conclude that every element of the form $Y = \sigma^e(X)$ for $X \in P$, $e \in \mathbb{N}$ has rank $rk_Q(Y) \leq \alpha$. It follows that each $0^n 1^\frown 0^\omega$ has rank $\leq \alpha + 1$ and that $rk_Q(0^\omega) \leq \alpha + 2$.

The computable embedding of P into Q was given by the mapping Φ where $\Phi(X) = (0^{X(0)}1^\frown 0^{X(0)+X(1)+1}1 \ldots)$ which then led us to construct $P \subseteq Q$. Certainly $D^{\alpha+1}(Q) = \{0^\omega, 0^n 1^\frown 0^\omega : n \in \mathbb{N}\}$ is the set of eventually periodic points of Q. Finally, for any $Y = \sigma^e(X)$ in Q where $e \in \mathbb{N}$, $X \in P$, Y has the same truth-table degree as $X \in P$.

To obtain $Q_1 \subseteq 2^{\mathbb{Z}}$, let $\Phi_1(X) = 0^{-\omega}.1^\frown\Phi(X)$. Then in part (0), we have again the shifts $\sigma^k(\Phi_1(X))$, now with $k \in \mathbb{Z}$. In part (1), we have all Z with exactly one occurence of 1, and part (2), we have 0^∞. □

Note that if P is perfect (that is, if $D(P) = P$), then Q is also perfect but we can still say that Φ maps the set of non-eventually-periodic points of Q onto P.

We obtain the following corollary to the proof of Theorem 4.6.

Corollary 4.7. *For any Π^0_1 class $P \subseteq 2^{\mathbb{N}}$, there is a Π^0_1 subshift $Q \subseteq 2^{\mathbb{N}}$ such that $Deg(Q) = Deg(P) \cup \{\mathbf{0}\}$.*

Note that in fact the mapping Φ in the proof of Theorem 4.6 is truth-table computable, so that Corollary 4.7 also applies to truth-table degrees.

The next corollary may now be obtained from standard results on Π^0_1 classes.

Corollary 4.8. *For any degree $\mathbf{b}$ such that either $\mathbf{b} \leq_T \mathbf{0}'$ or $\mathbf{0}' \leq_T \mathbf{b} \leq_T \mathbf{0}''$, there is a Π^0_1 subshift $Q \subseteq 2^{\mathbb{N}}$ of rank four (and also a Π^0_1 subshift $Q_1 \subseteq 2^{\mathbb{Z}}$) such that*

(i) Every element of Q (Q_1) of rank 2 or 3 is eventually periodic.

(ii) Every element of Q (Q_1) of rank 1 has Turing degree $\mathbf{b}$.

Proof. This corollary follows from Theorem 4.6 together with Corollary 3.2 of [5] and Theorem 2.1 of [11]. Given degree $\mathbf{b}$ as postulated, there exists (by the cited results) a Π^0_1 class P of rank two and a real X of degree $\mathbf{b}$ such that $D(P) = \{X\}$ and furthermore every other element of Y of P is eventually 0,that is, $Y = u^\frown 0^\omega$ for some u. Applying Theorem 4.6, we obtain a Π^0_1 subshift Q of rank four such that every element of rank ≥ 2 is eventually periodic. If Y has rank one in Q, then $Y = \sigma^e(X)$ for some e and hence Y has the same Turing degree as X. (Note that due to a different definition of rank in [5, 11], P is said to have rank one.) □

Note that those theorems do not apply to truth-table degrees and likewise this corollary does not hold for truth-table degrees.

We have now seen that many Δ^0_3 reals can belong to Π^0_1 subshifts of rank four. On the other hand, it is easy to see that every member of a Π^0_1 subshift of rank four is Δ^0_3.

Proposition 4.9. *For any Π^0_1 subshift Q of rank four, every element of Q is Δ^0_3.*

Proof. Let Q be a Π^0_1 subshift of rank four. Then $D^2(Q)$ has rank two, so that its members are all eventually periodic. Thus any element of rank two or three in Q is computable. The isolated members of Q are also computable. Finally, suppose that X has rank one in Q. Then X is isolated in the Π^0_3 class $D(Q)$ and is therefore Δ^0_3. □

On the other hand, an arbitrary Π^0_1 class of rank four may contain members which are not Σ^0_6 and even a Π^0_1 class of rank three may contain members which are not Σ^0_4.

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