Effective Randomness of Unions and Intersections

Douglas Cenzer
Department of Mathematics
P.O. Box 118105 University of Florida
Gainesville, FL 32611-8105
email: cenzer@ufl.edu,
Rebecca Weber
Department of Mathematics
Dartmouth College

June 6, 2012

Abstract

We investigate the μ-randomness of unions and intersections of random sets under various notions of randomness corresponding to different probability measures. For example, the union of two relatively Martin-Löf random sets is not Martin-Löf random but is random with respect to the Bernoulli measure $\lambda_{\frac{3}{4}}$ under which any number belongs to the set with probability $\frac{3}{4}$. Conversely, any $\lambda_{\frac{3}{4}}$ random set is the union of two Martin-Löf random sets. Unions and intersections of random closed sets are also studied.

1 Introduction

The study of algorithmic randomness has been of great interest in recent years. The basic problem is to quantify the randomness of a single real number. Early in the last century, von Mises [11] suggested that a random real should have reasonable statistical properties, such as the proportion of ones on the first n bits limiting to $\frac{1}{2}$. Thus a random real would be *stochastic* in modern parlance.

Martin-Löf [7] observed that stochastic properties could be viewed as special kinds of measure zero sets and defined a random real as one which avoids certain effectively presented measure 0 sets. That is, a real $X \in 2^{\mathbb{N}}$ is Martin-Löf random with respect to probability measure μ if for any effective sequence $S_1, S_2, \dots$ of c.e. open sets with $\mu(S_n) \leq 2^{-n}$, $X \notin \cap_n S_n$.

Research was partially supported by the National Science Foundation grants DMS 0532644 and 0554841 and 0062393. Thanks also to the American Institute of Mathematics for support during 2006 Effective Randomness Workshop.

Keywords: Computable analysis, computability, randomness

In this paper we want to consider the interaction of notions of algorithmic randomness corresponding to different probability measures, both for infinite binary sequences and for closed subsets of the Cantor space. Martin-Löf randomness for closed sets was defined in a recent paper [1] and is described below. We are particularly interested in the Bernoulli measure λ_p (with $0 < p < 1$) where the probability of a "1 "is p and the probability of a "0 "is $1-p$. The real X is μ-random *relative to* A (or $A - \mu$ *random*) if it meets all Martin-Löf μ-tests which are uniformly c.e. relative to A.

Van Lambalgen's Theorem is a fundamental result of algorithmic randomness which shows that the join $A \oplus B$ of two subsets of $\mathbb{N}$ is Martin-Löf random if and only if A is Martin-Löf random relative to B and B is Martin-Löf random relative to A. This theorem has many applications; in particular, it was used in [2] to show that every Martin-Löf random closed set contains a Martin-Löf random element.

We now consider versions of Van Lambalgen's Theorem for unions and intersections of random sets. It is easy to see that the union of two relatively Martin-Löf random sets is not Martin-Löf random under the standard Lebesgue measure; however, the union *is* random with respect to the Bernoulli measure $\lambda_{\frac{3}{4}}$ under which any number belongs to the set with probability $\frac{3}{4}$. Conversely, any $\lambda_{\frac{3}{4}}$ random set is the union of two Martin-Löf random sets. Unions and intersections of random closed sets are also studied.

Some definitions are needed. For a finite string $\sigma \in \{0,1\}^n$, let $|\sigma| = n$. For two strings σ, τ, say that τ extends σ and write $\sigma \prec \tau$ if $|\sigma| \leq |\tau|$ and $\sigma(i) = \tau(i)$ for $i < |\sigma|$. Similarly $\sigma \prec x$ for $x \in 2^{\mathbb{N}}$ means that $\sigma(i) = x(i)$ for $i < |\sigma|$. Let $\sigma^\frown\tau$ denote the concatenation of σ and τ and let $\sigma^\frown i$ denote $\sigma^\frown(i)$ for $i = 0, 1$. For any $\sigma \in \{0,1\}^*$ and any $x \in 2^{\mathbb{N}}$, let $\sigma^\frown x = (\sigma(0), \ldots, \sigma(|\sigma|-1), x(0), x(1), \ldots)$. Let $x\lceil n = (x(0), \ldots, x(n-1))$. Two reals x and y may be coded together into the join $z = x \oplus y$, where $z(2n) = x(n)$ and $z(2n+1) = y(n)$ for all n. We normally identify a set $A \subseteq \mathbb{N}$ with its characteristic function in $2^{\mathbb{N}}$.

For a finite string σ, let $I[\sigma]$ denote $\{x \in 2^{\mathbb{N}} : \sigma \prec x\}$. We shall call $I[\sigma]$ the *interval* determined by σ. The clopen sets are exactly the finite unions of intervals. A nonempty closed set P may be identified with a tree $T_P \subseteq \{0,1\}^*$ where $T_P = \{\sigma : P \cap I(\sigma) \neq \emptyset\}$. Note that T_P has no dead ends. That is, if $\sigma \in T_P$, then either $\sigma^\frown 0 \in T_P$ or $\sigma^\frown 1 \in T_P$.

For an arbitrary tree $T \subseteq \{0,1\}^*$, let $[T]$ denote the set of infinite paths through T, that is,

$$x \in [T] \iff (\forall n) x\lceil n \in T.$$

It is well-known that $P \subseteq 2^{\mathbb{N}}$ is a closed set if and only if $P = [T]$ for some tree T. The set P is a Π^0_1 class, or effectively closed set, if $P = [T]$ for some computable tree T. The set P is a strong Π^0_2 class if $P = [T]$ for some Δ^0_2 tree. The complement of a Π^0_1 class is sometimes called a c.e. open set. We remark that if P is a Π^0_1 class, then T_P is a Π^0_1 set, but it is not, in general, computable. There is a natural effective enumeration $P_0, P_1, \ldots$ of the Π^0_1 classes and thus an enumeration of the c.e. open sets. Thus we can say that a sequence $S_0, S_1, \ldots$

of c.e. open sets is *effective* if there is a computable function, f, such that $S_n = 2^{\mathbb{N}} - P_{f(n)}$ for all n. For a detailed development of Π^0_1 classes, see [4, 5].

The disjoint union $P \oplus Q$ of two closed sets is $\{0^\frown X : X \in P\} \cup \{1^\frown Y : Y \in Q\}$. The product $P \otimes Q$ is $\{X \oplus Y : X \in P \;\&\; Y \in Q\}$.

A tree T with no dead ends may be represented (or coded) as an element $x = x_T$ of $3^{\mathbb{N}}$, where the nodes of T are addressed in lexicographic order as $\tau_0, \tau_1, \dots$ and the value of the bit corresponding to σ is interpreted as follows:

(0) If $x(n) = 0$, then $\tau_n{}^\frown 0 \in T$ and $\tau_n{}^\frown 1 \notin T$.

(1) If $x(n) = 1$, then $\tau_n{}^\frown 0 \notin T$ and $\tau_n{}^\frown 1 \in T$.

(2) If $x(n) = 2$, then $\tau_n{}^\frown 0 \in T$ and $\tau_n{}^\frown 1 \in T$.

Thus the tree T may be produced from $x = x_T \in 3^{\mathbb{N}}$ by the following process. Begin by using $x(0)$ to determine whether one or both of the extensions (0) and (1) of the root τ_0 are in T; this will define τ_1 and possibly τ_2. Then use $x(1)$ to determine the extensions of τ_1 and continue in this fashion. For example, if x starts with (1210), then we will have $\tau_0 = \emptyset$, $\tau_1 = (1)$, $\tau_2 = (10)$, $\tau_3 = (11)$, $\tau_4 = (101)$, and $\tau_5 = (110)$. In particular, $x(0) = 1$ means that T will have the node (1) but will not have the node (0).

It is clear that this process defines a map taking $x = x_T$ to T which is a computable, one-to-one function from $3^{\mathbb{N}}$ onto the set of trees with no dead ends. We use the standard notion of computable functions on $\mathbb{N}^{\mathbb{N}}$ as given for example in Chapter 3 of Soare [9].

If Q is a closed set with corresponding tree $T_Q = T$, having no dead ends, and $x = x_T$, then we also write $x = x_Q$ and say that x is the *canonical code* for Q. It is clear that this defines an effective one-to-one mapping from the space $3^{\mathbb{N}}$ onto the space $\mathcal{C}$ of closed subsets of $2^{\mathbb{N}}$.

The standard (*hit-or-miss*) topology on the space $\mathcal{C}$ of closed sets is given by a sub-basis of sets of two types, where U is any open set in $2^{\mathbb{N}}$.

$$V(U) = \{K : K \cap U \neq \emptyset\}; \qquad W(U) = \{K : K \subseteq U\}$$

Note that $W(\emptyset) = \{\emptyset\}$ and that $V(2^{\mathbb{N}}) = \mathcal{C} \setminus \{\emptyset\}$, so that $\emptyset$ is an isolated element of $\mathcal{C}$ under this topology. Thus we may omit $\emptyset$ from $\mathcal{C}$ without complications. See [3] for details.

For our space $\mathcal{C}$, there is a simpler basis of clopen sets of the form

$$U_A = \{K : (\forall \sigma \in \{0,1\}^n) : \sigma \in A \iff K \cap I[\sigma] \neq \emptyset\},$$

where $A \subseteq \{0,1\}^n$. It is easy to see that there is a simple sub-basis of sets, obtained by taking, for each string $\sigma \in \{0,1\}^*$, the set $V(I[\sigma])$. In particular, because we assume that the choices of branching from each node are mutually independent, this means that any measure on the space is determined by its values on these sub-basic sets.

The definition of a random (nonempty) closed set $P = [T]$ by Brodhead, Cenzer and Dashti [1] comes from a probability measure where, for every node

$\sigma \in T$, each of the scenarios above has equal probability $\frac{1}{3}$. The closed set $P = [T]$ is random if its code x_T is Martin-Löf random in $3^{\mathbb{N}}$ with respect to the standard Lebesgue measure λ on $3^{\mathbb{N}}$, which assigns probability $\frac{1}{3}$ to each choice of 0, 1, or 2. There is a strong Π^0_2 closed set which is Martin-Löf random but no Π^0_1 class is Martin-Löf random. It was shown that a random closed set is perfect and contains no computable elements (in fact, it contains no n-c.e. elements). Every random closed set has measure 0 and has box dimension $\log_2 \frac{4}{3}$.

For any positive reals p, q such that $p + q \leq 1$, we can define a Bernoulli probability measure $\lambda_{\langle p,q\rangle}$ on $3^{\mathbb{N}}$ by having $x(n) = 0$ with probability p, $x(n) = 1$ with probability q and $x(n) = 2$ with probability $1-p-q$. The induced measure on the space $\mathcal{C}$ of closed subsets of $2^{\mathbb{N}}$ is denoted by μ^*, where

$$\mu^*(\mathcal{X}) = \mu(\{x_Q : Q \in \mathcal{X}\}). \tag{1}$$

In the case that $q_0 = q_1$, we say that μ^* is *symmetric*.

For our purposes, we will sometimes use the equivalent formulation of randomness for closed sets via *ghost codes* [2]. Here we simply enumerate *all* strings from $\{0,1\}^*$ as $\tau_0, \tau_1, \ldots$ and, if $\tau_n \in T_P$, we use $x(n) \in \{0, 1, 2\}$ to determine the branching below τ_n. Thus if $\tau_n \notin T$, then the value $x(n)$ is not used in the definition of T. Thus it may be thought of as a ghost code.

In general, x is used to determine whether a given string τ is in the corresponding tree T as follows. For each $i \leq |\tau|$, let n_i be the unique n such that $\tau\lceil i = \tau_n$. Then $\tau \in T$ if and only if, for each $i \leq |\tau|$, we have the following. If $\tau(i) = 0$, then either $x(i) = 0$ or $x(i) = 2$ and if $\tau(i) = 1$, then either $x(i) = 1$ or $x(i) = 2$.

For example, consider x which begins with (1210) as above but now as a ghost code representation for a closed set. Then we have the standard enumeration $\tau_0 = \emptyset$, $\tau_1 = (0)$, $\tau_2 = (1)$ and $\tau_3 = (00)$. Once again, the value $x(0) = 1$ means that $(1) \in T$ but $(0) \notin T$. However, the value $x(1) = 2$ applies to $\tau_1 = (0)$ and thus has no effect on the defintion of T. The value $x(2) = 1$ now applies to $\tau_2 = (1)$ and thus puts $(11) \in T$ and $(10) \notin T$. Finally, the value $x(3) = 0$ applies to $\tau_3 = (00)$ which is not in T and hence $x(3)$ has no effect on the definition of T.

The definition of ghost codes means that any tree T has infinitely many different representations via ghost codes. The connection between the original representation and the ghost code representation is given by Theorem 2.4 of [2], which states that a closed set $Q \subset 2^{\mathbb{N}}$ is Martin-Löf random if and only if there is *some* ghost code representation $x \in 3^{\mathbb{N}}$ which is Martin-Löf random

We will want to consider more general measures on both $\Sigma^{\mathbb{N}}$ and on $\mathcal{C}$. A measure μ on $\Sigma^{\mathbb{N}}$ is determined by its values on the intervals $I[\sigma]$ and is said to be computable if there is an algorithm which computes the measure $\mu(I[\sigma])$ from input σ. For any sequence $\alpha = (a_0, a_1, \ldots)$ where $0 \leq a_i \leq 1$ for all i, we may define the measure λ_α on $2^{\mathbb{N}}$ by setting $\lambda_\alpha(I[\sigma]) = \prod_{i<|\sigma|} b_i$, where $b_i = a_i$ if $\sigma(i) = 1$ and $b_i = 1 - a_i$ if $\sigma(i) = 0$. Note that the probability of n belonging to an arbitrary set $A \in 2^{\mathbb{N}}$ is independent of the probabilities of any other numbers being in A. We will say that a probability measure with this property is *generalized Bernoulli.* If α is computable, then we say that λ_α is a computable measure. Any computable generalized Bernoulli measure λ on $2^{\mathbb{N}}$ may be determined by some computable sequence α as above. An even more general notion of a computable measure could define the probability that $n \in A$ as a function of n as well as $A\lceil n$.

If λ and μ are two measures on $2^{\mathbb{N}}$, the product measure $\lambda \otimes \mu$ on $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is defined by setting $(\lambda \otimes \mu)(U \times V) = \lambda(U) \cdot \mu(V)$. When $\lambda = \lambda_\alpha$ and $\mu = \lambda_\beta$, then $\lambda \otimes \mu$ may be identified with $\lambda_{\alpha \oplus \beta}$ in the following sense. Let $\psi : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the natural homeomorphism with $\psi(A, B) = A \oplus B$. Then for any subset U of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, $(\lambda_\alpha \otimes \lambda_\beta)(U) = \lambda_{\alpha \oplus \beta}(\psi(U))$.

We will also consider the product $\alpha \cdot \beta = (a_0 \cdot b_0, a_1 \cdot b_1, \dots)$ of two sequences and the corresponding measure $\lambda_{\alpha \cdot \beta}$.

We will use the following lemma to transform Martin-Löf tests on one space to Martin-Löf tests on another space. Here we say that $(\mathcal{X}, \mu)$ is a computable probability space if $\mathcal{X} = \Sigma^{\mathbb{N}}$ for some finite set Σ and μ is a computable measure on $\mathcal{X}$ such that $\mu(\mathcal{X}) = 1$.

Lemma 1.1. *Let $(\mathcal{X}, \mu)$ be a computable probability space and let $F : \mathcal{X} \to \mathcal{Y}$ be a computable function and let $\mathcal{Y}$ be a space of the form $\Sigma^{\mathbb{N}}$ for some finite Σ. Define the measure ν on $\mathcal{Y}$ by $\nu(U) = \mu(F^{-1}(U))$.*

1. *ν is a computable probability measure on $\mathcal{Y}$.*
2. *If $\{V_e : e \in \mathbb{N}\}$ is a ν - Martin-Löf test on $\mathcal{X}$, then the sequence $\{F^{-1}(V_e) : e \in \mathbb{N}\}$ is a μ - Martin-Löf test on $\mathcal{Y}$.*
3. *If A is μ - Martin-Löf random, then $F(A)$ is ν - Martin-Löf random.*

Proof. It is easy to see that ν is a probability measure on $\mathcal{Y}$. Since μ is computable, there is an algorithm to compute $\mu(I[\sigma])$ from σ for $\sigma \in \Sigma^*$. Since F is computable, there is an algorithm which computes the inverse image $F^{-1}(I[\tau])$ as a finite union of intervals in $\mathcal{X}$. Combining these, we see that $\nu(I[\tau])$ may be computed from τ, so that ν is computable.

Since F is computable and the sequence $\{V_e : e \in \mathbb{N}\}$ is uniformly c.e., it follows that $\{F^{-1}(V_e) : e \in \mathbb{N}\}$ is also uniformly c.e.. For each e, $\nu(F^{-1}(V_e)) = \mu(V_e) \leq 2^{-e}$, so that $\{F^{-1}(V_e) : e \in \mathbb{N}\}$ is a Martin-Löf test.

Finally, suppose that A is μ - Martin-Löf random and let $B = F(A)$. Let $U_1, U_2, \dots$ be a ν - Martin-Löf test for B so that $\nu(U_e) \leq 2^{-e}$ for all e and let $V_e = F^{-1}(U_e)$. Since F is computable and $\{U_e : e \in \mathbb{N}\}$ is uniformly c.e., it follows that $\{V_e : e \in \mathbb{N}\}$ is also uniformly c.e.. For all e, $\mu(V_e) = \nu(U_e) \leq 2^{-e}$, so that $\{V_e : e \in \mathbb{N}\}$ is a μ -Martin-Löf test for A. Since A is μ - Martin-Löf random, there is some e such that $A \notin V_e$ and hence $B \notin U_e$. Thus B passes the arbitrary ν - Martin-Löf test and is therefore ν - Martin-Löf random. □

Here is an application of this lemma, a part of van Lambalgen's Theorem [10]. (The full result will be proved below).

Proposition 1.2. *Let λ_1 and λ_2 be two computable probability measures on $2^{\mathbb{N}}$ and let λ be the product measure. If $A \oplus B$ is λ - Martin-Löf random, then B is λ_2 - Martin-Löf random.*

Proof. Let $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be defined by $F(X \oplus Y) = Y$. Since $X \oplus Y$ is λ -Martin-Löf random, the result will follow if we can show that $\lambda_2(U) = \lambda(F^{-1}(U))$ for each open set U. □

We shall make frequent use of the following version of van Lambalgen's Theorem [10].

Theorem 1.3. *Let λ_1 and λ_2 be two computable measures on $2^{\mathbb{N}}$ and let $\lambda = \lambda_1 \otimes \lambda_2$ be the product measure. If A is λ_1-random relative to B and B is λ_2-random, then $A \oplus B$ is λ-random.*

Proof. Suppose that $A \oplus B$ is not $\lambda_1 \otimes \lambda_2$-random and let $A \oplus B \in \bigcap_n V_n$ for some Martin-Löf test with $\lambda(V_n) \leq 2^{-2n}$ for each n.

For each n, let

$$F_n(Y) = \lambda_1(\{X : X \oplus Y \in V_n\})$$

and define a *Solovay* Martin-Löf λ_2-test by

$$W_n = \{Y : F_n(Y) > 2^{-n}\}.$$

Then we have the following calculation.

$$\lambda(V_n) = \iint_{V_n} dY dX = \int_Y F_n(Y) dY \geq \int_{W_n} F_n(Y) dY \geq \lambda_2(W_n) \cdot 2^{-n}$$

Since $\lambda(V_n) \leq 2^{-2n}$, it follows that $\lambda_2(W_n) \leq 2^{-n}$. Since $W_{n+1} \subseteq W_n$ for each n, this is a Solovay test.

Since B is λ_2-random, it follows that $B \notin W_n$ for almost all n. By renumbering we may assume that $B \notin W_n$ for any n.

Now let $U_n = \{X : X \oplus B \in V_n\}$. It follows that $\lambda_1(U_n) \leq 2^{-n}$ for all n and this is a λ_1 - Martin-Löf test relative to B.

Since A is λ_1 - Martin-Löf random relative to B, it follows that $A \notin U_n$ for some n. But this means that $A \oplus B \notin V_n$, contradicting the initial assumption. □

Here is the converse theorem.

Theorem 1.4. *Let λ_1 and λ_2 be two computable probability measures on $2^{\mathbb{N}}$ and let $\lambda = \lambda_1 \otimes \lambda_2$ be the product measure. If $A \oplus B$ is λ-random, then A is λ_1-random relative to B and B is λ_2-random relative to A.*

Proof. By symmetry, it suffices to show that A is λ_1-random relative to B. We proceed by the contrapositive.

We first construct a universal oracle λ_1 Martin-Löf test U_b^Y. We begin by constructing a uniformly c.e. sequence $V_{e,k}^Y$ of all possible λ_1 Martin-Löf tests as follows. As usual, let $W_e^Y = \{i : \phi_e^Y(i) \downarrow\}$ where we assume that for each s and Y, there is at most one i such that $i \in W_{e,s+1}^Y \setminus W_{e,s}^Y$. Then define the uniformly c.e. sequence of sets $G_{e,k}^Y$ in stages $G_{e,k,s}^Y$ where $i \in G_{e,k,s+1}^Y$ if and only if $\langle i,k\rangle \in G_{e,k,s}^Y$ or $\langle i,k\rangle \in W_{e,s+1}^Y \setminus W_{e,s}^Y$ and $\lambda_1(I[\sigma_i] \cup \bigcup\{I[\sigma_j] : j \in G_{e,k,s}^Y\}) \leq 2^{-k}$. Now let $V_{e,k}^Y = \bigcup\{I[\sigma_j] : j \in G_{e,k}^Y\}$. Then $\lambda_1(V_{e,k}^Y) \leq 2^{-k}$ for all e, k.

It follows that each $\{V_{e,k}^Y : k \in \mathbb{N}\}$ is a λ_1 - Martin-Löf test. Now suppose that $\{G_k : k \in \mathbb{N}\}$ is some λ_1 - Martin-Löf test. Then it is uniformly c.e., so there is some e such that, for all k, $G_k = \bigcup\{I[\sigma_i] : \langle i,k\rangle \in W_e\}$ and it follows from the construction that $G_k^Y = V_{e,k}^Y$.

Finally, let $U_e^Y = \bigcup_n V_{n+e+1}^Y$. Then

$$\lambda_1(U_e^Y) \leq \sum_n \lambda_1(V_{n+e+1}^Y) \leq \sum_e 2^{-n-e-1} = 2^{-e}.$$

Now suppose that A is *not* λ_1 random relative to B. Then $A \in \bigcap_e U_e^B$. Let $S_e = \{X \oplus Y : X \in U_e^Y\}$. Then $\{S_e : e \in \mathbb{N}\}$ is uniformly c.e. and we will show that $\lambda(S_e) \leq 2^{-e}$ for all e. Let $F_e : 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ be the characteristic function of S_e. Then, for each e,

$$\lambda(S_e) = \iint_{2^{\mathbb{N}} \times 2^{\mathbb{N}}} F_e(X,Y) = \int_Y \int_X F_e(X,Y) = \int_Y \lambda_1(U_e^Y) \leq \int_Y 2^{-e} \leq 2^{-e}.$$

Since $A \in \bigcap_e U_e^B$, for each e, it follows that $A \oplus B \in S_e$ and hence $A \oplus B$ is *not* λ - Martin-Löf random. □

2 Unions and Intersections of Random Sets

Suppose that A and B are relatively Martin-Löf random. By van Lambalgen's Theorem, $A \oplus B$ is Martin-Löf random. However, $A \cup B$ is *not* random, since it has asymptotic density $\frac{3}{4}$. Likewise, $A \cap B$ is not random, since it has density $\frac{1}{4}$. We will show that $A \cap B$ is $\lambda_{\frac{1}{4}}$-random and $A \cup B$ is $\lambda_{\frac{3}{4}}$-random.

This is a consequence of the following more general result. For an infinite sequence $\alpha = (a_0, a_1, \dots)$ of reals with $0 \leq a_i \leq 1$ for all i, let $1 - \alpha = (1 - a_0, 1 - a_1, \dots)$

Theorem 2.1. *Let $\alpha = (a_0, a_1, \dots)$ and $\beta = (b_0, b_1, \dots)$ be two infinite sequences with $0 \leq a_i \leq 1$ and $0 \leq b_i \leq 1$ for each i. If A is λ_α-random relative to B and B is λ_β-random relative to A, then $A \cap B$ is $\lambda_{\alpha\cdot\beta}$-random and $A \cup B$ is $\lambda_{1-(1-\alpha)\cdot(1-\beta)}$-random.*

Proof. Let A and B be relatively random, so $A \oplus B$ is random. Define the computable function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $F(C) = C_0 \cap C_1$, where we may recall that $C_i = \{X : i^\frown X \in C\}$.) Then $F(A \oplus B) = A \cap B$.

Lemma 2.2. *For any open $G \subseteq 2^{\mathbb{N}}$, $\lambda_{\alpha\cdot\beta}(G) = \lambda_{\alpha\oplus\beta}(F^{-1}(G))$.*

Proof. It suffices to prove this for sets of the form $G = \{Y : Y(i) = 1\}$, so fix i and let $G = \{Y : Y(i) = 1\}$. For this G, $\lambda_{\alpha\cdot\beta}(G) = a_i \cdot b_i$. Now $F^{-1}(G) = \{C : i \in C_0 \cap C_1\} = \{C : 2i \in C\} \cap \{2i+1 \in C\}$. Since $\alpha \oplus \beta = (a_0, b_0, a_1, b_1, \dots)$, it follows that $\lambda_{\alpha\oplus\beta}(F^{-1}(G)) = a_i \cdot b_i$. □

It now follows from Lemma 1.1 that $A \cap B$ is $\lambda_{\alpha\cdot\beta}$-random.

To see that $A \cup B$ is $\lambda_{1-(1-\alpha)\cdot(1-\beta)}$ random, observe first that $\mathbb{N} - A$ is $\lambda_{1-\alpha}$ random and $\mathbb{N} - B$ is $\lambda_{1-\beta}$ random. Then $\mathbb{N} - (A \cup B) = (\mathbb{N} - A) \cap (\mathbb{N} - B)$ is $\lambda_{(1-\alpha)(1-\beta)}$ random and the result follows. □

Next we consider a converse result.

Theorem 2.3. *Let $\alpha = (a_0, a_1, \dots)$ and $\beta = (b_0, b_1, \dots)$ be two infinite sequences with $0 \le a_i \le 1$ and $0 \le b_i \le 1$ for each i.*

(i) If C is $\lambda_{\alpha\cdot\beta}$-random, then there exist A and B such that $C = A \cap B$, A is λ_α-random relative to B and B is λ_β-random relative to A.

(ii) If C is $\lambda_{1-(1-\alpha)\oplus(1-\beta)}$-random, then there exist A and B such that $C = A \cup B$, A is λ_α-random and B is λ_β-random.

Proof. Suppose that C is $\lambda_{\alpha\cdot\beta}$-random. For each i, let $p_i = \frac{a_i(1-b_i)}{1-a_ib_i}$, $q_i = \frac{b_i(1-a_i)}{1-a_ib_i}$, and $r_i = 1 - p_i - q_i = \frac{(1-a_i)(1-b_i)}{1-a_ib_i}$ and let $\gamma_i = (p_i, q_i, r_i)$ be a probability sequence on $\{0,1,2\}^{\mathbb{N}}$. Now let $g \in \{0,1,2\}^{\mathbb{N}}$ be γ-random. Define A and B as follows.

$$i \in A \iff i \in C \ \vee\ g(i) = 0;$$
$$i \in B \iff i \in C \ \vee\ g(i) = 1;$$

It is clear that $C = A \cap B$. It remains to show that A is λ_α random and that B is λ_β random. It follows as in Theorem 1.3 that $C \oplus g$ is random with respect to the product measure $\lambda_{(\alpha\cdot\beta)\oplus\gamma}$.

Define the computable map $F : 2^{\mathbb{N}} \otimes \{0,1,2\}^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$F(X \oplus f) = (X \cup \{i : f(i) = 0\}) \oplus (X \cup \{i : f(i) = 1\}).$$

Note that $F(C \oplus g) = A \oplus B$. We will show that $A \oplus B$ is $\lambda_{\alpha\oplus\beta}$ random and hence A is λ_α random relative to B and B is λ_β random relative to A by Theorem 1.4. To show this, we may apply Lemma 1.1 to conclude that $A \oplus B$ is random with respect to the measure λ defined by $\lambda(U) = \lambda_{(\alpha\cdot\beta)\oplus\gamma}(F^{-1}(U))$. Thus it remains to prove the following lemma.

Lemma 2.4. *For any open set U, $\lambda_{(\alpha\cdot\beta)\oplus\gamma}(F^{-1}(U)) = \lambda_{\alpha\oplus\beta}(U)$.*

Proof. It suffices to show the result for sub-basic open sets of the form $U = \{Y : Y(i) = 0\}$. Fix $n \in \mathbb{N}$ and let $U = \{Y : Y(n) = 0\}$. There are two cases to consider, depending on whether n is even or odd.

Suppose first that $n = 2i$ is even. Then the probability $\lambda_{\alpha\oplus\beta}(U) = 1 - a_i$.

Now $X \oplus f \in F^{-1}(U)$ if and only if $F(X \oplus f) \in U$, that is, iff $n \notin (X \cup \{i : f(i) = 0\}) \oplus (X \cup \{i : f(i) = 1\})$. Since $n = 2i$, this is equivalent to

$i \notin X \cup \{i : f(i) = 0\}$, which is equivalent to $i \notin X$ and $f(i) \neq 0$. By the definition of $\alpha \cdot \beta$, $i \notin X$ with probability $1 - a_i b_i$. By the definition of γ, $f(i) \neq 0$ with probability $1 - p_i$. Thus

$$\lambda(F^{-1}(U)) = (1 - a_i b_i)(1 - p_i) = (1 - a_i b_i)(1 - \frac{a_i(1 - b_i)}{1 - a_i b_i})$$
$$= 1 - a_i b_i - a_i + a_i b_i = 1 - a_i = \lambda_{\alpha \cdot \beta}(U).$$

The second case, when $n = 2i + 1$ is even, has a similar proof. □

This completes the proof of part (i) of the theorem, as outlined above.
Part (ii) follows as in the proof of Theorem 2.1. □

We want to consider how effective this proof is. Observe that in the proof, we may take f to be Δ^0_2 in C, so that both A and B can be Δ^0_2 in C.

Problem 2.5. *Given, say, a $\lambda_{3/4}$ random set C, must there exist Martin-Löf random sets A, B with $C = A \cup B$ such that A, B are in fact computable in C, or c.e. in C? or $\leq_{LR} C$? (Here $A \leq_{LR} C$ means that any real which is Martin-Löf random relative to C is also Martin-Löf random relative to A.)*

3 Unions and Intersections of Random Closed Sets

In this section, we consider the randomness of unions and intersections of closed sets. These are two independent problems here since the complement of a closed set is not closed.

As described in the introduction, we may define a probability sequence $\alpha = (\langle p_0, q_0\rangle, \langle p_1, q_1\rangle, \dots)$ on $\mathcal{C}$ where the branching below the string τ_n is dictated by the probability (p_n, q_n); that is, τ_n has unique extension $\tau_n{}^\frown 0$ with probability p_n, τ_n has unique extension $\tau_n{}^\frown 1$ with probability q_n and τ_n has both extensions with probability $1 - p_n - q_n$.

There is a version of van Lambalgen's Theorem for disjoint unions of closed sets in [2]. Here we give a more general version.

Theorem 3.1. *Let $\alpha = (\langle p_0, q_0\rangle, \langle p_1, q_1\rangle, \dots)$ and $\beta = (\langle r_0, s_0\rangle, \langle r_1, s_1\rangle, \dots)$ be two infinite sequences with $p_i, q_i, r_i, s_i \geq 0$, $0 \leq p_i + q_i \leq 1$ and $0 \leq r_i + s_i \leq 1$ for each i. If P is λ^*_α-random relative to Q and Q is λ^*_β-random relative to P, then $P \oplus Q$ is $(\lambda_\alpha * \lambda_\beta)^*$-random, where $(\lambda_\alpha * \lambda_\beta)^*$ is defined so that $x(0) = 2$ with probability 1, so that the branching at $0^\frown\tau_i$ is determined by $\langle p_i, q_i\rangle$ and the branching at $1^\frown\tau_i$ is determined by $\langle r_i, s_i\rangle$. Conversely, if $P \oplus Q$ is $(\lambda_\alpha * \lambda_\beta)^*$-random, then P is λ^*_α-random relative to Q and Q is λ^*_β-random relative to P.*

Proof. Let P have *ghost code* $X = (x_0, x_1, \dots)$ and Q have ghost code $Y = (y_0, y_1, \dots)$ which are random as described. Then $P \oplus Q$ has ghost code

$$Z = F(X,Y) = (z_0, z_1, \dots) = (2, x_0, y_0, x_1, x_2, y_1, y_2, x_3, x_4, x_5, x_6, x_7, y_4, \dots).$$

That is, in general, the codes $x_{2^n-1}, x_{2^n}, \dots, x_{2^{n+1}-2}$ which determined the branching in P of the (ghost) nodes of length n will determine in $P \oplus Q$ the branching of the nodes of length $n+1$ which begin with 0 and similarly the codes $y_{2^n-1}, y_{2^n}, \dots, y_{2^{n+1}-2}$ which determined the branching in Q of the (ghost) nodes of length n will determine in $P \oplus Q$ the branching of the nodes of length $n+1$ which begin with 1. Observe that the function $F : \{0,1,2\}^{\mathbb{N}} \otimes \{0,1,2\}^{\mathbb{N}} \to \{0,1,2\}^{\mathbb{N}}$ is computable and one-to-one.

Lemma 3.2. *Let the measure λ on $3^{\mathbb{N}}$ be defined by setting $\lambda(U) = (\lambda_\alpha * \lambda_\beta)(F^{-1}(U))$. Then for any X and any Y such that X is λ_α-random relative to Y and Y is λ_β-random relative to X, $Z = F(X,Y)$ is $(\lambda_\alpha * \lambda_\beta)$-random.*

Proof. This follows immediately from the computability of F. □

Now $P \oplus Q$ has ghost code $Z = F(X,Y)$ and Z is $(\lambda_\alpha * \lambda_\beta)$ - random, so that $P \oplus Q$ is $(\lambda_\alpha * \lambda_\beta)^*$ random.

For the converse, the $(\lambda_\alpha * \lambda_\beta)$-randomness of Z implies that the code X of P is λ_α-random relative to the code Y of Q and similarly Y is λ_β-random relative to X. □

Next we consider the randomness of the product $P \otimes Q$ of two closed sets. Given $\alpha = (\langle p_0, q_0\rangle, \langle p_1, q_1\rangle, \dots)$ and $\beta = (\langle r_0, s_0\rangle, \langle r_1, s_1\rangle, \dots)$ such that $p_i, q_i, r_1, s_i \geq 0$, $0 \leq p_i + q_i \leq 1$ and $0 \leq r_i + s_i \leq 1$ for each i with corresponding measures λ_α and λ_β, define the measure $(\lambda_\alpha \otimes \lambda_\beta)^*$ in two cases as follows.

First let $\sigma = (i_0, j_0, \dots, i_{n-1}, j_{n-1})$ be a string of even length. Then the branching at σ is determined by the branching of $(i_0, \dots, i_{n-1})$ under the probability measure λ_α.

Second, let $\sigma = (i_0, j_0, \dots, i_{n-1}, j_{n-1}, i_n)$ be a string of odd length. Then the branching at σ is determined by the branching of $(j_0, \dots, j_{n-1})$ under the probability measure λ_β.

We also need the notion of a projection for closed sets. For any $A \in 2^{\mathbb{N}}$, define the projections $\pi_0(A) = A_0 = A(0), A(2), \dots)$ and $\pi_1(A) = A_1 = (A(1), A(3), \dots)$; then $A = \pi_0(A) \oplus \pi_1(A)$. For a set $R \subseteq 2^{\mathbb{N}}$ and for $i = 0, 1$, let $\pi_i(R) = \{\pi_i(A) : A \in R\}$ be the projections of the set R. These set mappings are computable. If $R = P \otimes Q$ for some sets P and Q, then $P = \pi_0(R)$ and $Q = \pi_1(R)$.

Theorem 3.3. *Let $\alpha = (\langle p_0, q_0\rangle, \langle p_1, q_1\rangle, \dots)$ and $\beta = (\langle r_0, s_0\rangle, \langle r_1, s_1\rangle, \dots)$ be two infinite sequences with $p_i, q_i, r_i, s_i \geq 0$, $0 \leq p_i + q_i \leq 1$ and $0 \leq r_i + s_i \leq 1$ for each i. Then P is λ_α^*- Martin-Löf random relative to Q and Q is λ_β^*- Martin-Löf random relative to P if and only if $P \otimes Q$ is $(\lambda_\alpha \otimes \lambda_\beta)^*$- Martin-Löf random.*

Proof. Suppose that P is λ_α^*-random relative to Q and Q is λ_β^*-random relative to P, so that $P \oplus Q$ is $(\lambda_\alpha \oplus \lambda_\beta)^*$ - random.

Define the function F so that $F(R) = R_0 \otimes R_1$, so that $F(P \oplus Q) = P \otimes Q$. Then by Lemma 1.1, $P \oplus Q$ is random with respect to the measure λ^* given by $\lambda^*(V) = \lambda_{\alpha\oplus\beta}^*(F^{-1}(V))$.

Lemma 3.4. *For any Borel set $V \subseteq \mathcal{C}$, $(\lambda_\alpha \otimes \lambda_\beta)^*(V) = (\lambda_\alpha * \lambda_\beta)^*(F^{-1}(V))$.*

Proof. This is easy to check. For example, let $V = V(I[(01)]) = \{R : R \cap I[(01)] \neq \emptyset\}$. Then $(\lambda_\alpha \otimes \lambda_\beta)^*(V) = (1 - q_0)(1 - r_0)$.

$R \in F^{-1}(V)$ if and only if $R_0 \otimes R_1$ meets $I[(01)]$ which is if and only if R_0 meets $I[(0)]$) and R_1 meets $I[(1)]$, which is if and only if $(01) \in T_R$. So under the measure $(\lambda_\alpha \otimes \lambda_\beta)^*(F^{-1}(V)) = (1 - q_0)(1 - r_0)$ as well. □

We have to work a little harder for the converse, since F is not a surjection, although it is an injection. We proceed by the contrapositive. Suppose that $P \oplus Q$ is *not* Martin-Löf random with respect to the measure $(\lambda_\alpha * \lambda_\beta)^*$ and let $\{V_0, V_1, \dots\}$ be a $(\lambda_\alpha * \lambda_\beta)^*$ Martin-Löf test such that, for every n, $(\lambda_\alpha * \lambda_\beta)^*(V_n) \leq 2^{-n}$ and $P \oplus Q \in V_n$.

Let $W_n = F[V_n] = \{F(R) : R \in V_n\}$; since F is a computable injection, then this will be a uniformly c.e. sequence of open sets. Certainly $F(P \oplus Q) = P \otimes Q \in W_n$ for every n. Since F is one-to-one, it follows that $V_n = F^{-1}(W_n)$ and hence by Lemma 3.4, we have

$$(\lambda_\alpha \otimes \lambda_\beta)^*(W_n) = (\lambda_\alpha * \lambda_\beta)^*(V_n) \leq 2^{-n}.$$

But this implies that $P \otimes Q$ is *not* $(\lambda_\alpha \otimes \lambda_\beta)^*$ - Martin-Löf random, as desired. □

We note here that under the measures $\lambda_{\langle p,q\rangle}^*$, where $p + q > 0$, the class of closed sets which are products has measure zero. To see this, just observe that in a product $R = P \otimes Q$, it can never be the case that $T_R \cap \{0,1\}^2 = \{(00), (11)\}$ nor may it equal $\{(01), (10)\}$. For the measure λ_p^*, this immediately shows that the class of products has measure $\leq \frac{7}{9}$. Consideration of $T_R \cap \{0,1\}^{2n}$ will show that the measure is $\leq (\frac{7}{9})^n$ and thus the set of products has measure zero. This restriction contrasts with the fact that any closed set may be expressed as a union and also as an intersection.

Determining the appropriate randomness for unions and intersections is rather complicated, so that we will only consider only Bernoulli measures here, where the branching probabilities are the same for all nodes. Suppose that p and q are non-negative real numbers such that $p + q \leq 1$. Then we define the measure $\lambda_{\langle p,q\rangle}$ so that, for any $n \in \mathbb{N}$ and any $X \in 3^{\mathbb{N}}$, the probability that $X(n) = 0$ is p, the probability that $X(n) = 1$ is q, and therefore the probability that $X(n) = 2$ is $1 - p - q$. Thus for the corresponding measure $\lambda_{\langle p,q\rangle}^*$ on $\mathcal{C}$, we for any $Q \in \mathcal{C}$ and any $\sigma \in T_Q$, σ will have the unique extension $\sigma^\frown 0$ in T_Q with probability p, will have the unique extension $\sigma^\frown 1$ in T_Q with probability q, and will have both extensions with probability $1 - p - q$. (Recall that $\sigma \in T_Q$ if and only if $Q \cap I[\sigma] \neq \emptyset$.) In the case that $p = q$, we will abbreviate $\lambda_{\langle p,p\rangle}$ as λ_p.

We first consider the algorithmic randomness of unions of closed sets.

Theorem 3.5. *Suppose that $p, q, r, s \geq 0$, $0 \leq p + q \leq 1$ and $0 \leq r + s \leq 1$. Suppose that the closed set P is $\lambda^*_{\langle p,q\rangle}$-random relative to Q and that Q is $\lambda^*_{\langle r,s\rangle}$-random relative to P. Then $P \cup Q$ is λ^*-random for a certain measure λ. For the special case when $p = q = r = s$, the probability $\lambda^*(I[\sigma])$ that $P \cup Q$ meets an interval $I[\sigma]$ when $|\sigma| = n$ equals $(1-p)^n(2-(1-p)^n)$.*

Proof. Let $\lambda = \lambda_{\langle p,q\rangle} * \lambda_{\langle r,s\rangle}$, so that if P is $\lambda^*_{\langle p,q\rangle}$ Martin-Löf random and Q is $\lambda^*_{\langle r,s\rangle}$ Martin-Löf random, then $P \oplus Q$ will be $(\lambda_{\langle p,q\rangle} * \lambda_{\langle r,s\rangle})^*$ random. Define the function $F : \mathcal{C} \to \mathcal{C}$ by $F(C) = C_0 \cup C_1$, where $C_i = \{x : i^\frown x \in C\}$. This means that $F(P \oplus Q) = P \cup Q$. Then by Lemma 1.1, $P \cup Q$ is random with respect to the induced measure λ^* defined by $\lambda^*(V) = (\lambda_{\langle p,q\rangle} * \lambda_{\langle r,s\rangle})^*(F^{-1}(V))$.

Now we will show how to compute the measure λ. First let $f_0(\sigma)$ be the probability that an arbitrary closed set P meets $I[\sigma]$ under the measure $\lambda^*_{\langle p,q\rangle}$. Then $f_0(\sigma) = \prod_{i<|\sigma|} m_i$, where $m_i = 1 - q$ if $\sigma(i) = 0$ and $m_i = 1 - p$ if $\sigma(i) = 1$. Similarly let $f_1(\sigma)$ be the probability that Q meets $I[\sigma]$ under the measure $\lambda^*_{\langle r,s\rangle}$. Finally, define $g(\sigma) = \lambda^*(I[\sigma])$.

Consider first the probability that, for arbitrary closed sets P and Q, $(0) \in T_{P\cup Q}$, that is $\lambda^*(\{R : (R_0 \cup R_1) \cap I[(0)] \neq \emptyset\}$. This event occurs when $x_P(0) \neq 1$ and $x_Q(0) \neq 1$ and thus has probability $g((0)) = 1 - qs$. Similarly $g((1)) = 1 - pr$.

Now let $\tau = 0^\frown\sigma$. Then $g(\tau)$ may be calculated in three cases.

First, with probability $(1-q)(1-s)$, we may have $P \cap I[((0))] \neq \emptyset$ and $Q \cap I[((0))] \neq \emptyset$, and then $\tau \in T_{P\cup Q}$ with relative probability $g(\sigma)$.

Second, with probability $(1-q)s$, we may have $P \cap I[((0))] \neq \emptyset$ and $Q \cap I[((0))] = \emptyset$, and then $\tau \in x_{P\cup Q}$ with relative probability $f_0(\sigma)$.

Third, with probability $q(1-s)$, we may have $P \cap I[((0))] = \emptyset$ and $Q \cap I[((0))] \neq \emptyset$, and then $\tau \in x_{P\cup Q}$ with relative probability $f_1(\sigma)$.

It follows that

$$g(0^\frown\sigma) = (1-q)(1-s)g(\sigma) + (1-q)sf_0(\sigma) + q(1-s)f_1(\sigma).$$

Similarly, we will have

$$g(1^\frown\sigma) = (1-p)(1-r)g(\sigma) + (1-p)rf_0(\sigma) + p(1-r)f_1(\sigma).$$

The desired measure λ is now determined by the values $\lambda^*(I[\sigma]) = g(\sigma)$. This demonstrates the following.

Lemma 3.6. *For any Borel set $V \subseteq \mathcal{C}$, $\lambda^*(V) = (\mu * \nu)^*(F^{-1}(V))$.*

It follows that $P \cup Q$ is λ^*-random, as desired.

For the special case when $p = q = r = s$, we have $f_0(\sigma) = f_1(\sigma) = (1-p)^{|\sigma|}$ for all σ and we have $g((i)) = 1 - p^2$ for $i = 0, 1$. Letting $G(n) = g(\sigma)$, where σ is any string of length n, we have $G(1) = 1 - p^2$ and obtain the recursive formula

$$G(n+1) = (1-p)^2 G(n) + 2p(1-p)^{n+1}.$$

It may be seen by induction that $G(n) = (1-p)^n(2-(1-p)^n)$. In particular, for $p = \frac{1}{3}$, we have $G(n) = (\frac{2}{3})^n(2-(\frac{2}{3})^n)$. □

In particular, for $p = \frac{1}{3}$, this implies that for the original measure λ from [2], if P is random relative to Q and Q is random relative to P, then $P \cup Q$ is λ^* -random, where in general the probability $\lambda^*(I[\sigma])$ that $P \cup Q$ meets an interval $I[\sigma]$ when $|\sigma| = n$ equals $(\frac{2}{3})^n(2 - (\frac{2}{3})^n)$.

We pose the question of whether this result can be *inverted* as with the results for random sets of natural numbers.

Problem 3.7. *Suppose that* $0 \leq p \leq \frac{1}{2}$. *If* R *is* λ^* *Martin-Löf random, where the measure* λ *is determined by having the probability* $\lambda^*(I[\sigma])$ *of meeting an interval* $I[\sigma]$, *when* $|\sigma| = n$, *equal to* $(1-p)^n(2-(1-p)^n)$, *do there exist* P *and* Q *such that* $R = P \cup Q$, P *is* λ_p^**-random relative to* Q *and* Q *is* λ_p^**-random relative to* P*?*

We next consider intersections of random closed sets. This becomes rather interesting because the intersection of two random closed sets may of course be empty. That is, for any Bernoulli measure λ and any λ^* Martin-Löf random closed set P, it follows from the coding that the closed set $0^\frown P$ is also λ^* Martin-Löf random, where $0^\frown P = \{0^\frown x : x \in P\}$. Then if P and Q are relatively random, the closed sets $0^\frown P$ and $1^\frown Q$ will be relatively random and will be disjoint.

Theorem 3.8. *Suppose that* $p, q, r, s \geq 0$, $0 \leq p + q \leq 1$ *and* $0 \leq r + s \leq 1$. *Suppose that the closed set* P *is* $\lambda^*_{\langle p,q\rangle}$ *- Martin-Löf random relative to* Q *and that* Q *is* $\lambda^*_{\langle r,s\rangle}$ *- Martin-Löf random relative to* P. *Then we have the following*

1. *If* $p+q+r+s \geq 1+pr+qs$, *then* $P \cap Q$ *is always empty.*

2. *If* $p+q+r+s < 1+pr+qs$, *then* $P \cap Q$ *is empty with probability* $e = \frac{ps+qr}{(1-p-q)(1-r-s)}$

3. *If* $p+q+r+s < 1+pr+qs$ *and* $P \cap Q$ *is nonempty, then* $P \cap Q$ *is Martin-Löf random with respect to the measure* $\lambda^*_{\langle p+r-pr, q+s-qs\rangle}$.

Proof. Let e be the probability that $P \cap Q = \emptyset$. It follows that for any node $\sigma \in T_P \cap T_Q$, the probability that σ has an infinite extension in $P \cap Q$ is also e. Considering the nine possible cases for the initial branching of P and of Q, we obtain the equation

$$e = (ps+qr) + (1-p-q)(1-r-s)e^2 + (p+q+r+s-pr-qs-2ps-2qr)e.$$

There are two cases where $P_1 \cap Q_1 = \emptyset$ where one branches only to the left and the other branches only to the right. This has probability $ps + qr$.

There is one case where both P and Q have both branches. This has probability $(1-p-q)(1-r-s)$.

There are six cases where $P \cap Q$ has exactly one branch with a total probability of $(p+q+r+s-pr-qs-2ps-2qr)$.

This equation has two possible solutions, $e = 1$ and $e = \frac{ps+qr}{(1-p-q)(1-r-s)}$.

Now we can prove the three parts of our theorem.

(1) If $p+q+r+s \geq 1+pr+qs$, then $\frac{ps+qr}{(1-p-q)(1-r-s)} \geq 1$ and hence we must have $e=1$, so that $P \cap Q$ is empty with probability one.

Now $\{P : P_0 \cap P_1 = \emptyset\}$ is a c.e. open set. It follows that the complement is a Π^0_1 closed subset of $\mathcal{C}$ with measure zero, so that no Martin-Löf random closed set can belong to it. Hence for any random closed set P, $P_0 \cap P_1 = \emptyset$.

(2) If $p+q+r+s < 1+pr+qs$, then $\frac{ps+qr}{(1-p-q)(1-r-s)} < 1$ and this will be the probability e that $P \cap Q$ is empty. To see this, consider the trees T_P and T_Q and let e_n be the probability that T_P and T_Q have no common nodes of length n. The reasoning above tells us that

$$e_{n+1} = (ps+qr) + (1-p-q)(1-r-s)e_n^2 + (p+q+r+s-pr-qs-2ps-2qr)e_n.$$

Then e is the limit of the increasing sequence $\langle e_n \rangle_n$ and it can be seen that $e_n \leq \frac{ps+qr}{(1-p-q)(1-r-s)}$ for all n, so that $e = \lim_n e_n = \frac{ps+qr}{(1-p-q)(1-r-s)}$.

(3) Finally, suppose that $P \cap Q$ *is* nonempty. It remains to calculate the branching probabilities *relative to this assumption.*

Assuming that $\sigma \in T_{P\cap Q}$, there are seven possible cases for whether $\sigma^\frown 0$ and/or $\sigma^\frown 1$ are in either of T_P or T_Q.

There is one possible case where both $\sigma^\frown 0$ and $\sigma^\frown 1$ are in $T_{P\cap Q}$. Both of them must be in $T_P \cap T_Q$ and also each branch must be nonempty. This occurs with probability $(1-p-q)(1-r-s)(1-e)^2$.

There are four cases where $T_{P\cap Q}$ has only the left branch.

First, both branches are in $T_P \cap T_Q$, with only the left branch being nonempty. This has probability $(1-p-q)(1-r-s)e(1-e) = (ps+qr)(1-e)$.

Second, only the left branch is in $T_P \cap T_Q$ and it is nonempty. This has probability $pr(1-e)$.

Third, T_P has only the left branch, T_Q has both branches, and the left branch is nonempty. This has probability $p(1-r-s)(1-e)$.

Fourth, T_P has both branches, T_Q has only the left branch, and the left branch is nonempty. This has probability $(1-p-q)r(1-e)$. The total probability from these four cases is $(p+r-pr)(1-e)$.

Similarly, there are four cases where $P \cap Q$ has only the right branch and this has an extension in $P \cap Q$, with a total probability of $(q+s-qs)(1-e)$.

The remaining probability that neither branch has an extension in $P \cap Q$ is of course e but this can be disregarded since we have assumed that σ has an extension in $P \cap Q$.

It follows that the relative probability of having only the left branch $\sigma^\frown 0 \in T_{P\cap Q}$ is $p+r-pr$, the probability of having only a right branch is $q+s-qs$ and the remaining probability of having both branches is $1-p-q-r-s+pr+qs$.

Now define the function $F : \mathcal{C} \to \mathcal{C}$ by $F(K) = K_0 \cap K_1$. Then we have proved the following.

Lemma 3.9. *For any Borel set $V \subseteq \mathcal{C}$, $\lambda^*(V) = (\mu * \nu)^*(F^{-1}(V))$.*

Now given a λ^*-Martin-Löf test $\{U_n : n \in \mathbb{N}\}$ for $P \cap Q$, define $K = P \oplus Q$, so that $K_0 = P$ and $K_1 = Q$ and thus $F(K) = P \cap Q$. It follows from Theorem 3.1 that K is $(\mu * \nu)^*$-Martin-Löf random.

By Lemma 3.9, $\lambda^*(U_n) = (\mu * \nu)^*(F^{-1}(U_n))$. However, the map F is not computable, so that we must work a bit harder.

For each K, $F(K) = K_0 \cap K_1 \in F^{-1}(U(I[\sigma]))$ if and only if $K_0 \cap K_1 \cap I[\sigma] \neq \emptyset$, which is a Π^0_1 condition. Thus the sequence $\{F^{-1}(U_n) : n \in \mathbb{N}\}$ is not necessarily a Martin-Löf test.

Following the idea of Kjos-Hanssen and Diamondstone [6], we observe that with probability 1, if $K_0 \cap K_1 \cap I[\sigma] \neq \emptyset$, then $K_0 \cap K_1 \cap I[\sigma]$ is infinite. Since the probability e such that an intersection is empty is < 1, we can compute, for each n and ℓ, a value $m_{n,\ell}$ large enough such that $e^m \leq 2^{-n-2\ell}$. Let Φ be a partial computable functional such that $\Phi(n, \ell, K)$ is the least L such that, for all strings σ of length ℓ, either σ has no extension in $K_0 \cap K_1$ of length L, or σ has $\geq m_{n,\ell}$ extensions of length L, each of which has extensions in both K_0 and in K_1.

We claim that for each n and ℓ, $\Phi(n, \ell, K)$ is defined for almost all K. Fix n and ℓ and let $|\sigma| = \ell$. There are two possibilities. First, suppose that $K_0 \cap K_1 \cap I[\sigma] = \emptyset$. Then for some L_σ, σ has no extension in $K_0 \cap K_1$ of length L, so that $\Phi(n, k, \ell)$ is certainly defined. Second, suppose that $K_0 \cap K_1 \cap I[\sigma] \neq \emptyset$. Then by the remarks above, we may assume that $K_0 \cap K_1 \cap I[\sigma]$ is infinite. Thus for some L_σ, σ has $\geq L$ extensions in $K_0 \cap K_1$ of length L and therefore $\Phi(n, \ell, K)$ exists and is $\leq \max\{L_\sigma : |\sigma| = \ell\}$. Since $\{K : \Phi(n, \ell, K) \downarrow\}$ is a c.e. open set of measure 1, $\Phi(n, \ell, K)$ exists for every Martin-Löf random K.

Let $K(\ell) = \bigcup\{I[\sigma] : |\sigma| = \ell \ \& \ K_0 \cap K_1 \cap I[\sigma] \neq \emptyset\}$ and define the approximation $K(\ell, L)$ to be the union of the intervals $I[\sigma]$ such that $|\sigma| = \ell$ and σ has an extension τ of length L such that both $K_0 \cap I[\tau] \neq \emptyset$ and $K_1 \cap I[\sigma] \neq \emptyset$. Let

$$V_n = \{K : (\exists \ell) K(\ell, \Phi(n, \ell, K)) \in U_n\}, \text{ and}$$

$$W_n = \{K : (\exists \ell) K(\ell, \Phi(n, \ell, K)) \neq K(\ell)\}.$$

Each of these sets is c.e. open. It follows from Lemma 3.9 that V_n has measure $< 2^{-n}$, since if $K \in V_n$ then $K_0 \cap K_1 \in U_n$.

Lemma 3.10. $(\mu * \nu)^*(W_n) \leq 2^{-n}$.

Proof. Suppose $K \in W_n$, so that $\Phi(n, \ell, K) = L$ and $K(\ell, L) \neq K(\ell)\}$. Let $m = m_{n,\ell}$. Then there is some σ of length ℓ such that σ has at least m extensions τ of length L such that $I[\tau]$ meets both K_0 andK_1, but $K_0 \cap K_1 \cap I[\tau] = \emptyset$. For each such τ, the probability that $K_0 \cap K_1 \cap I[\tau] = \emptyset$ is e and hence the combined probability is e^m that all are empty. By the choice of m, $e^m \leq 2^{-n-2\ell}$. Summing over the 2^ℓ possible choices of $\sigma \in \{0,1\}^\ell$, we obtain bound $2^{-n-\ell}$ for each ℓ. Finally, summing over ℓ, we obtain the desired upper bound 2^{-n}. □

Now $\{V_n \cup W_n : n \in \mathbb{N}\}$ is a Martin-Löf test. Since K is $(\mu * \nu)^*$ Martin-Löf random by assumption, there exists n such that $K \notin V_n \cup W_n$. Since $K \notin W_n$, $K(\ell, \Phi(n, \ell, K)) = K(\ell)$ for every ℓ. Since $K \notin V_n$, it follows that $K_0 \cap K_1 \notin U_n$. Thus $K_0 \cap K_1$ passes the Martin-Löf test and is λ^* Martin-Löf random, as desired. □

This can be applied to the special case of a symmetric measure ($p = q$ and $r = s$) and to the intersection of two $\lambda_{p,q}$ random closed sets.

First consider the symmetric case.

Corollary 3.11. *Let p and r be real numbers with $0 \leq p \leq \frac{1}{2}$ and $0 \leq r \leq \frac{1}{2}$. Let P be λ_p^*-random relative to Q and let Q be λ_r^*-random relative to P. Then we have the following.*

1. *If $2p + 2r \geq 1 + 2pr$, then $P \cap Q$ is always empty.*
2. *If $2p+2r < 1+2pr$, then $P \cap Q$ is empty with probability $e = \frac{2pr}{(1-2p)(1-2r)}$.*
3. *If $2p + 2r < 1 + 2pr$ and $P \cap Q$ is nonempty, then $P \cap Q$ is Martin-Löf random with respect to the measure given by $\langle p + r - pr, p + r - pr\rangle$.*

Second consider the case where $\langle p, q\rangle = \langle r, s\rangle$.

Corollary 3.12. *Let p and q be real numbers with $0 \leq p + q \leq 1$. Let P be $\lambda^*_{\langle p,q\rangle}$-random relative to Q and let Q be $\lambda^*_{\langle p,q\rangle}$-random relative to P. Then we have the following*

1. *If $2p + 2q \geq 1 + p^2 + q^2$, then $P \cap Q$ is always empty.*
2. *If $2p+2q < 1+p^2+q^2$, then $P \cap Q$ is empty with probability $e = \frac{2pq}{(1-p-q)^2}$.*
3. *If $2p+2q < 1+p^2+q^2$ and $P \cap Q$ is nonempty, then $P \cap Q$ is Martin-Löf random with respect to the measure given by $\langle 2p - p^2, 2q - q^2\rangle$.*

Finally, put the two together. For $p = q$, the equation $2p + 2q = 1 + p^2 + q^2$ becomes $4p = 1 + 2p^2$, which has solution $p = 1 - \frac{\sqrt{2}}{2}$. Thus we have the following.

Corollary 3.13. *Let p be a real number with $0 \leq p \leq \frac{1}{2}$. Let P be λ_p^*-random relative to Q and let Q be λ_p^*-random relative to P. Then we have the following*

1. *If $1 - \frac{\sqrt{2}}{2} < p \leq 1$, then $P \cap Q$ is always empty.*
2. *If $0 \leq p < 1 - \frac{\sqrt{2}}{2}$, then $P \cap Q$ is empty with probability $e = \frac{2p^2}{(1-2p)^2}$*
3. *If $p = 1 - \frac{\sqrt{2}}{2}$, and $P \cap Q$ is nonempty, then $P \cap Q$ is Martin-Löf random with respect to the measure given by $\langle 2p - p^2, 2p - p^2\rangle$.*

Thus for $p = \frac{1}{3}$ and the measure $\lambda^*_{\frac{1}{3}}$ as in [2], the intersection of any two relatively $\lambda^*_{\frac{1}{3}}$ -Martin-Löf random closed sets is empty, since $\frac{1}{3} > 1 - \frac{\sqrt{2}}{2}$. When $p = \frac{1}{4}$, the intersection of two relatively $\lambda^*_{\frac{1}{4}}$ -Martin-Löf random closed sets is empty with probability $\frac{1}{2}$ and, if nonempty, the intersection is $\lambda^*_{\frac{7}{16},\frac{7}{16}}$ - Martin-Löf random

This leads to the following natural question. If R is a $\lambda^*_{\frac{7}{16}}$ Martin-Löf random closed set, do there exist $\lambda^*_{\frac{1}{4}}$ Martin-Löf random closed sets P and Q such that $P \cap Q = R$?

More generally, we have the following.

Problem 3.14. *If p, q, r, s are all between 0 and 1, $p+q+r+s < 1+pr+qs$ and R is Martin-Löf random with respect to the measure given by $\lambda^*_{\langle p+r-pr, q+s-qs\rangle}$, do there exist P and Q such that $R = P \cap Q$, P is $\lambda_{\langle p,q\rangle}$ - Martin-Löf random and Q is $\lambda^*_{\langle r,s\rangle}$ -Martin-Löf random?*

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