Sample Questions for Test Two

1. Every element of an ordinal $\alpha$ is an ordina.

2. The class of ordinals is linearly ordered by $\in$.

3. If $\alpha$ is an ordinal, then $s(\alpha)$ is an ordinal.

4. For any two ordinals $\alpha \neq \beta$, $\alpha \in \beta$ if and only if $\alpha \subset \beta$.

5. For any nonempty class $A$ of ordinals, $\bigcap A$ is an ordinal.

6. For any set $A$ of ordinals, $\bigcup A$ is an ordinal.

7. Prove the Principle of Transfinite induction: Suppose that $\phi$ is a formula of set theory with parameters. Suppose that for every ordinal $\alpha$, $(\forall \beta \in \alpha \phi(\beta)) \rightarrow \phi(\alpha)$ holds. Then, for every ordinal $\alpha$, $\phi(\alpha)$ holds.

8. Prove the Principle of Set Induction: Suppose that $\phi$ is a formula of set theory with parameters. Suppose that, for every set $x$, $[(\forall y \in x)\phi(y)] \rightarrow \phi(x)$. Then $\phi(x)$ holds for every set $x$.

9. For any monotone class operator $\Gamma$, and any ordinals $\alpha \leq \beta$, $\Gamma^\alpha \subseteq \Gamma^\beta$.

10. For any ordinals $\alpha \leq \beta$, $V_\alpha \subseteq V_\beta$.

11. For any ordinal $\alpha$, $V_\alpha$ is transitive.

12. For every set $x$, there is an ordinal $\alpha$ such that $x \in V_\alpha$.

13. For each ordinal $\alpha$, $rk(\alpha) = \alpha$.

14. Show that if $P$ and $Q$ are closed, then $D(P \cup Q) = D(P) \cup D(Q)$.

15. Any properly decreasing chain of closed sets must be countable.

16. For any closed set $C$ of reals and any ordinals $\beta < \alpha$, $D^\alpha(C)$ is closed and $D^\alpha(C) \subseteq D^\beta(C)$.

17. Show that for any closed sets $P \subset Q$, $D(P) \subset D(Q)$.

18. Show that for any closed set $C$, $C \setminus D(C)$ is countable.

19. Ordinal addition is associative

20. For any ordinal $\alpha$ and any limit ordinal $\lambda$, $\alpha + \lambda$ is a limit ordinal

21. For any ordinals $\alpha \leq \beta$, there exists a unique $\delta$ such that $\beta = \alpha + \delta$

22. For any ordinals $\alpha$, $\beta$ and $\gamma$, if $\beta < \gamma$, then $\beta + \alpha \leq \gamma + \alpha$.

23. For any ordinals $\alpha$, $\beta$ and $\gamma$, if $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.

24. For all $\beta$, $0 \cdot \beta = 0$ and $1 \cdot \beta = \beta$. 
25. For all $\alpha, \beta, \gamma$, $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

26. Prove the Division Lemma: For all ordinals $\alpha$ and $\beta$, there exist ordinals $\rho < \alpha$ and $\delta$ such that $\beta = \alpha \cdot \delta + \rho$.

27. For any ordinals $\alpha$ and $\beta$, $(\alpha, \in) \oplus (\beta, \in)$ is isomorphic to $(\alpha + \beta, \in)$.

28. For any limit ordinal $\lambda$ and any ordinal $\alpha$, $\alpha \cdot \lambda$ and $\alpha + \lambda$ are limit ordinals.

29. Show that for any limit ordinal $\lambda$, there exists $\alpha$ such that $\lambda = \omega \cdot \alpha$.

30. Show that $2 \cdot \lambda = \lambda$ for any limit ordinal $\lambda$.

31. Every well-ordering is isomorphic to an ordinal.

32. For any countable limit ordinal $\lambda$, there is an infinite sequence $\beta_0 \leq \beta_1 \leq \ldots$ such that $\lambda = \bigcup \beta_i$ and there is an infinite sequence $\alpha_0, \alpha_1, \ldots$ such that $\lambda = \sum \alpha_i$.

33. Prove that Zermelo’s Principle is equivalent to the Axiom of Choice.

34. Prove that the Relational Axiom of Choice is equivalent to the Multiplicative Principle.

35. Show that the Axiom of Choice is equivalent to the Well-Ordering Principle.

36. Show that the Axiom of Choice is equivalent to Zorn’s Lemma

37. Show that Hausdorff’s Maximal Principle implies Zorn’s Lemma

38. Show that Kuratowski’s Principle implies the Hausdorff Maximal Principle.

39. Use Zorn’s Lemma to prove that there is a maximal filter on any infinite set.

40. Show that the Axiom of Choice implies the Trichotomy Principle.

41. Show that the Mapping Principle implies the Well-Ordering Principle.

42. Show that the Trichotomy Principle implies the Mapping Principle.

43. Show that $\aleph_1$ is a regular cardinal.

44. For any infinite cardinal $\kappa$, the cardinal product $\kappa \cdot \kappa = \kappa$.

45. Show that for any limit ordinal $\lambda$, $V_\lambda$ satisfies the Pair Axiom.

46. Show that for any limit ordinal $\lambda$, $V_\lambda$ satisfies the Power Set Axiom.

47. Show that for any limit ordinal $\lambda$, $V_\lambda$ satisfies the Union Axiom.

48. Show that for any countable ordinal $\alpha > \omega$, $V_\alpha$ does not satisfy the Axiom of Replacement.

49. Show that $V_\alpha$ satisfies the Axiom of Separation for all ordinals $\alpha$.

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