Set Theory

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Preface

This book was developed over many years from class notes for a set theory course at the University of Florida. This course has been taught to advanced undergraduates as well as lower level graduate students. The notes have been used more than thirty times as the course has evolved from seminar-style towards a more traditional lecture.

Axiomatic set theory, along with logic, provides the foundation for higher mathematics. This book is focused on the axioms and how they are used to develop the universe of sets, including the integers, rational and real numbers, and transfinite ordinal and cardinal numbers. There is an effort to connect set theory with the mathematics of the real numbers. There are details on various formulations and applications of the Axiom of Choice. Several special topics are covered. The rationals and the reals are studied as dense linear orderings without end points. The possible types of well-ordered subsets of the rationals and reals are examined. The possible cardinality of sets of reals is studied. The Cantor space $2^{\mathbb{N}}$ and Baire space $\mathbb{N}^{\mathbb{N}}$ are presented as topological spaces. Ordinal arithmetic is developed in great detail. The topic of the possible models of fragments of the axioms is examined. As part of the material on the axioms of set theory, we consider models of various subsets of the axioms, as an introduction to consistency and independence. Another interesting topic we cover is an introduction to Ramsey Theory.

It is reasonable to covermost of the material in a one semester course, with selective omissions. Chapter 2 is a review of sets and logic, and should be covered as needed in one or two weeks. Chapter 3 introduces the Axioms of Zermelo-Fraenkel, as well as the Axiom of Choice, in about two weeks. Chapter 4 develops the Natural Numbers, induction and recursion, and introduces cardinality, taking two or three weeks. Chapter 5 on Ordinal Numbers includes transfinite induction and recursion, well-ordinals, and ordinal arithmetic, in two or three weeks. Chapter 6 covers equivalent versions and applications of the Axiom of Choice, as well as Cardinality, in about two or three weeks. The Real Numbers are developed in Chapter 7, with discussion of dense and complete orders, countable and uncountable sets of reals, and a brief introduction to topological spaces such as the Baire space and Cantor space, again in two or three weeks. If all goes well, this leaves about one week each for the final two chapters: Models of set theory and an introduction to Ramsey theory.

The book contains nearly 300 exercises which test the students understanding and also enhance the material.

The authors have enjoyed teaching from these notes and are very pleased to share them with a broader audience.

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Chapter 1

Introduction

Set theory and mathematical logic compose the foundation of pure mathematics. Using the axioms of set theory, we can construct our universe of discourse, beginning with the natural numbers, moving on with sets and functions over the natural numbers, integers, rationals and real numbers, and eventually developing the transfinite ordinal and cardinal numbers. Mathematical logic provides the language of higher mathematics which allows one to frame the definitions, lemmas, theorems and conjectures which form the every day work of mathematicians. The axioms and rules of deduction set up the system in which we can prove our conjectures, thus turning them into theorems.

Chapter 2 begins with elementary naive set theory, including the algebra of sets under union, intersection, and complement and their connection with elementary logic. This chapter introduces the notions of relations, functions, equivalence relations, orderings, and trees. The fundamental notion is *membership*, that is, one set x being a member or element of a second set y; this is written $x \in y$. Then one set x is a subset of another set y, written $x \subseteq y$, if every element of x is also an element of y.

Chapter 3 introduces the axioms of Zermelo Fraenkel. A set should be determined by its elements. Thus the Axiom of Extensionality states that two sets are equal if and only if they contain exactly the same elements. Some basic axioms provide the existence of simple sets. For example, the Empty Set Axiom asserts the existence of the set $\emptyset$ with no elements. The Axiom of Pairing provides for any two sets x and y a set $\{x, y\}$ with exactly the two members x and y. The Union Axiom provides the union $x \cup y$ of any two given sets, as well as the more general union $\bigcup A$ of a family A of sets. With these we can create sets with three or more elements, for example $\{a, b, c\} = \{a, b\} \cup \{b, c\}$. The Powerset Axiom collects together into one set $\mathcal{P}(A)$ all subsets of a given set A. The Axiom of Infinity postulates the existence of an infinite set, and thus provides for the existence of the set $\mathbb{N}$ of natural numbers. The Axiom of Comprehension provides the existence of the definable subset $\{x \in A : P(x)\}$ of elements of a given set A which satisfy a property P. For example, given the set $\mathbb{N}$ of natural numbers, we can define the set of even numbers as $\{x : (\exists y)x = y + y\}$. The Axiom of Replacement provides the existence of the image $F[A]$ of a given set A under a definable function. The somewhat controversial Axiom of Choice states that for any family $\{A_i : i \in I\}$ of nonempty sets, there is a function F with domain I such that $F(i) \in A_i$ for all $i \in I$. This might seems to be an obvious fact, but it has very strong consequences. In particular, the Axiom of Choice implies the Well-Ordering Principle that every set can be well-ordered. A well-ordering $\lhd$ of a set A is an ordering with no descending sequences $a_1 \rhd a_2 \rhd \dots$. So the integers can be well-ordered by $0 \lhd 1 \lhd -1 \lhd 2 \lhd \dots$. However, any attempt to well order the set of real numbers will reveal that this is not so obvious after all. Finally, the Axiom of Regularity states that every set A contains a $\in$-minimal element, that is, a set $x \in A$ such that, for all $y \in A$, $y \notin x$. In particular, this implies that no set can belong to itself, and therefore there can be no universal set of all sets. The Axiom of Regularity implies that there is no chain of sets $A_0, A_1, \dots$ such that $A_{n+1} \in A_n$ for all n. This principle is needed to prove theorems by induction on sets, in the same way that the standard well-ordering on the natural numbers leads to the principle of induction.

Chapter 4 introduces the notion of cardinality, including finite versus infinite, and countable versus uncountable sets. We define the von Neumann natural numbers —$N = \{0, 1, 2, \dots\}$ in the context of set theory. The Induction Principle for natural numbers is established. The methods of recursive and inductive definability over the natural numbers are used to define operations including addition and multiplication on the natural numbers. These methods are also used to define the transitive closure of a set A as the closure of A under the *union* operator and to define the hereditarily finite sets as the closure of 0 under the powerset operator. The Schröder–Bernstein Theorem is presented, as well as Cantor's Theorem, which shows that the set of subsets of natural numbers is uncountable, and thus the set of reals is also uncountable.

Chapter 5 covers ordinal numbers and their connection with well-orderings.

The notions of recursive definitions and the principle of induction on the ordinals are developed. The hierarchy V_α of sets is developed and the notion of *rank* is defined. The standard operations of addition, multiplication and exponentiation of ordinal arithmetic are defined by transfinite recursion. Various properties of ordinal arithmetic, such as the commutative, associative and distributive laws are proved using transfinite induction. This culminates in the Cantor Normal Form Theorem. Well ordered subsets of the standard real ordering are studied. It is shown that every countable well ordering is isomorphic to a subset of the rationals, and that any well ordered set of reals is countable.

Chapter 6 is focused on cardinal numbers and the Axiom of Choice. Zorn's Lemma and the Well-Ordering Principle are shown to be equivalent to the Axiom of Choice. Zorn's Lemma is used to prove the Prime Ideal Theorem and to show that every vector space has a basis. Cardinal numbers are defined and it is shown that, under the Axiom of Choice, every set has a unique cardinality. Hartog's Lemma, that every cardinal number has a successor, is proved, thus establishing the existence of uncountable cardinals. The operations of cardinal arithmetic are defined. The Continuum Hypothesis, that the reals have cardinality $\aleph_1$, is formulated. It is shown that the reals cannot have cardinality $\aleph_\omega$. The notion of cofinality and regular cardinals are defined, as well as weakly and strongly inaccessible.

Chapter 7 makes the connection between set theory and the standard mathematical topics of algebra, analysis, and topology. The integers, rationals, and real numbers are constructed inside of the universe of sets, starting from the natural numbers. The rationals are characterized, up to isomorphism, as the unique countable dense linear order without end points. The reals are characterized, up to isomorphism, up as the unique complete dense order without end points containing a countable dense subset. The notions of accumulation point and point of condensation are discussed. There is a careful proof of the Cantor–Bendixson theorem, that every closed set of reals can be expressed as a disjoint union of a countable set and a perfect closed set. There is a brief introduction to topological spaces. The Cantor space $2^{\mathbb{N}}$ and Baire space $\mathbb{N}^{\mathbb{N}}$ are studied. It is shown that a subset of $2^{\mathbb{N}}$ is closed if and only if it can be represented as the set of infinite paths through a tree.

Chapter 8 introduces the notion of a *model of set theory*. Conditions are given under which a given set A can satisfy certain of the axioms, such as the union axiom, the power set axiom, and so on. It is shown that the hereditarily finite sets satisfy all axioms except for the Axiom of Infinity.

The topic of the possible models of fragments of the axioms is examined. In particular, we consider the axioms that are satisfied by V_α when α is for example a limit cardinal, or an inaccessible cardinal. The hereditarily finite and hereditarily countable, and more generally hereditarily $< \kappa$ sets are also studied in this regard. The hereditarily finite sets are shown to satisfy all axioms except Regularity. This culminates in the proof that V_κ is a model of ZF if and only if κ is a strongly incaccessilble cardinal.

Chapter 9 is a brief introduction to Ramsey theory, which studies partitions. This begins with some finite versions of Ramsey's theorem and related results. There is a proof of Ramsey's Theorem for the natural numbers as well as the Erdös-Rado Theorem for pairs. Uncountable partitions are also studied.

This additional material gives the instructor options for creating a course which provides the basic elements of set theory and logic, as well as making a solid connection with many other areas of mathematics.