Sample Problem Solutions for Exam Two

1. Compute the Wronskian of $y_1 = x^2, y_2 = x^2 ln(x)$; are y_1, y_2 independent?
$W[y_1, y_2] = y_1 y_2' - y_1' y_2 = x^2(2x\ ln\ x + x) - 2x(x^2 ln\ x) = x^3$.
y_1 and y_2 are independent since the Wronskian is not identically zero.

2. Solve $y'' - 4y' + 5y = 0$ with y(0) = 3 and $y'(0) = 9$.
$r^2 - 4r + 5$ has roots $r = 2 \pm i$, so that
$y = e^{2x}(c_1 cos\ x + c_2 sin\ x)$
Then $3 = y(0) = c_1$ and $9 = y'(0) = 2c_1 + c_2$ so
$c_1 = 3$ and $c_2 = 3$ and $y = 3e^{2x}(cos\ x + sin\ x)$.

3. Use Undetermined Coefficients to solve $y'' + 4y = 6x + 4cos\ 2x$.
$y_h = c_1 cos\ 2x + c_2 sin\ 2x$ and
$y_p = Ax + B + Cx\ cos\ 2x + Dx\ sin\ 2x$.
$y_p'' = C(-4sin\ 2x - 4x\ cos\ 2x) + D(4cos\ 2x - 4x\ sin\ 2x)$,
$y_p'' + 4y_p = 4Ax + 4B - 4Csin\ 2x + 4Dcos\ 2x = 6x + 4cos\ 2x$,
so
$4A = 6$, $4B = 0$, $-4C = 0$, $4D = 4$ and $A = 1.5$, $B = C = 0$, $D = 1$.
$y_p = 1.5x + xsin\ 2x$ and $y = y_h + y_p = c_1 cos\ 2x + c_2 sin\ 2x + 1.5x + xsin\ 2x$

4. Use Variation of Parameters to solve $y'' - 4y' = e^{3x}$.
$r^2 - 4r = 0$ has roots $r = 0$ and $r = 4$, so $y_1 = 1$ and $y_2 = e^{4x}$ and $W[y_1, y_2] = 4e^{4x}$.
$y_p = u_1 y_1 + u_2 y_2 = u_1 + u_2 e^{4x}$, where
$u_1' = -f(x)y_2/W = -e^{3x}e^{4x}/4e^{4x} = -e^{3x}/4$, so $u_1 = -e^{3x}/12$.
$u_2' = f(x)y_1/W = e^{3x}/4e^{4x} = e^{-x}/4$, so $u_2 = -e^{-x}/4$.
Then $y_p = -e^{3x}/12 + (-e^{-x}/4)e^{4x}) = -e^{3x}/3$.

5. Find the general solution of the Euler equation $x^2 y'' + xy' - 4y = x^2 + 1$, given that $y_h = c_1 x^{-2} + c_2 x^2$.
$W = x^{-2}(2x) - (2x^{-3}(x^2) = 4x^{-1}$.
$F(x) = 1 + x^{-2}$ (dividing by x^2).
$y_p = v_1 x^{-2} + v_2 x^2$, where
$v_1' = -x^2(1 + x^{-2})/4x^{-1} = -x^3/4 - x/4$, so $v_1 = -x^4/16 - x^2/8$.
$v_2' = x^{-2}(1 + x^{-2})/4x^{-1} = x^{-1}/4 + x^{-3}/4$, so $v_2 = \frac{1}{4} ln\ x - x^{-2}/8$.
Then $y_p = (-x^4/16 - x^2/8)x^{-2} + (\frac{1}{4} ln\ x - x^{-2}/8(x^2) = -\frac{1}{16}x^2 + \frac{1}{4}x^2 ln\ x - \frac{1}{4}$,
or just $y_p = \frac{1}{4}x^2 ln\ x - \frac{1}{4}$.
$y = y_h + y_p = c_1 x^2 + c_2 x^{-2} + (x^2 ln\ x - 1)/4$.
FOR THE HINT: $(r^2 - r) + r - 4 = r^2 - 4$, so $y_h = c_1 x^2 + c_2 x^{-2}$.
Solving by Undetermined Coefficients:
$y_p = A + Bx^2 ln\ x$, $y_p' = B(2xln\ x + x)$ and $y_p'' = B(2ln\ x + 3)$. Then $x^2 y'' + xy' - 4y = 4Bx^2 - 4A = x^2 + 1$, so $4B = 1$ and $-4A = 1$. Thus $A = -1/4$ and $B = 1/4$ so $y_p = (x^2 ln\ x - 1)/4$ and

6. A 16 pound weight is suspended from a spring with $k = 18$ pounds per foot.

(a) The weight is pulled down 3 inches from equilibrium and then struck upwards with initial speed 2 feet per second. Find the equation of motion and give the amplitude, period and phase shift. Sketch the solution.

(b) An outside force of $12sin(6t)$ is applied to the spring at equilibrium. Write the differential equation and solve for $x(t)$. Find the equation of motion. What phenomenon does this represent?

(a) $m = 16/32 = .5$ so the differential equation is $.5x'' + 18x = 0$ with solution $x = c_1 cos\ 6t + c_2 sin\ 6t$.

Then $.25 = x(0) = c_1$ and $-2 = x'(0) = 6c_2$, so
$x = \frac{1}{4}cos\ 6t - \frac{1}{3}sin\ 6t$.
The period is $2\pi/6$ and the amplitude is $\sqrt{1/16 + 1/9} = 5/12$.
$x = \frac{5}{12}[\frac{3}{5}cos\ 6t - \frac{4}{5}sin\ 6t] = \frac{5}{12}[sin\ \theta cos\ 6t + cos\ \theta sin\ 6t] = \frac{5}{12}sin(6t + \theta)$,
where $sin\ \theta = \frac{3}{5}$ and $cos\ \theta = -\frac{4}{5}$, so $tan\ \theta = -\frac{3}{4}$.
The phase angle θ is approximately 2.5

(b) $.5x'' + 18x = 12sin\ 6t$ has particular solution of the form
$x_p = t(Acos\ 6t + Bsin\ 6t)$.
Then $x_p'' = -36t(Acos\ 6t + Bsin\ 6t) + 12(Bcos\ 6t - Asin\ 6t)$,
so $.5x'' + 18x = 6(Bcos\ 6t - Asin\ 6t) = 12sin\ 6t$,
making $B = 0$ and $A = -2$. Thus
$x_p = -2tcos\ 6t$ and
$x = -2tcos\ 6t + c_1 cos\ 6t + c_2 sin\ 6t$.
Equilibrium means $x(0) = x'(0) = 0$, so
$0 = x(0) = c_1$ and $0 = x'(0) = -2 + 6c_2$,
making $c_1 = 0$ and $c_2 = 1/3$. Thus
$x = \frac{1}{3}sin\ 6t - 2tcos\ 6t$.
This represents undamped resonance.

7. A mass of 4 grams is suspended from a spring with constant 16 dynes per centimeter. The mass is pulled down .5 cm and an external force of $12sin(t)$ is applied. Assuming a damping factor of $8v$ (dynes), write the differential equation and solve for $x(t)$. What is the steady state solution.

$4x'' + 8x' + 16x = 12sin\ t$ is the differential equation.
The corresponding polynomial is $r^2 + 2r + 4$ with roots $r = -1 \pm i\sqrt{3}$ so we have
$x_h = e^{-t}(c_1 cos\sqrt{3}t + c_2 sin\sqrt{3}t)$ and
$x_p = Acos\ t + Bsin\ t$
Then $x'' + 2x' + 4x = (3A + 2B)cos\ t + (3B - 2A)sin\ t = 3sin\ t$,
so $3A + 2B = 0$ and $3B - 2A = 3$ giving $A = -6/13$ and $B = 9/13$.
The steady state solution is
$x_p = \frac{9}{13}sin\ t - \frac{6}{13}cos\ t$.
Now $x = \frac{9}{13}sin\ t - \frac{6}{13}cos\ t + e^{-t}(c_1 cos\sqrt{3}t + c_2 sin\sqrt{3}t)$,
with $.5 = x(0) = -6/13 + c_1$ and $0 = x'(0) = \frac{9}{13} - c_1 + \sqrt{3}c_2$,
so $c_1 = 25/26$ and $c_2 = 7/26\sqrt{3}$.

8. Solve the system $x' = x - 4y; \quad y' = x + y$

By the second equation, $x = y' - y$, so $x' = y'' - y'$.

Putting $x = y' - y$ into equation one, we get $x' = y' - y - 4y = y' - 5y$.

Then $y'' - y' = y' - 5y$, so $y'' - 2y' + 5y = 0$.

This means $y = e^t(c_1 cos\ 2t + c_2 sin\ 2t$.

Then $y' = e^t(c_1 cos\ 2t + c_2 sin\ 2t - 2c_1 sin\ 2t + 2c_2 cos\ 2t)$,

so $x = y' - y = e^t(-2c_1 sin\ 2t + 2c_2 cos\ 2t)$.

9. Factor the differential equation $y'' - 3y' + 2y = x$ into two first order equations and solve.

$(D-1)(D-2)y = (D-1)z = x$, where $z = (D-2)y$. Then

$z = e^x \int xe^{-x} = e^x(-x-1)e^{-x} = -x-1$, so

$y = e^{2x} \int (-x-1)e^{-2x} = e^{2x}(\frac{1}{2}x + \frac{3}{4})e^{-2x} = \frac{1}{2}x + \frac{3}{4}x$.

10. Use reduction of order (by the Wronskian and Abel's identity) to find a second solution to $xy'' + (1-2x)y' + (x-1)y = 0$ given that $y = e^x$ is one solution.

$p = x^{-1} - 2$, so $W = e^{2x - ln\ x} = x^{-1}e^{2x}$ and then

$$y_2 = e^x \int x^{-1}e^{2x}/(e^x)^2 = e^x \int x^{-1} = e^x ln\ x$$

11. Find the general solution of $y''' - 3y' + 2y = 0$.

$r^3 - 3r + 2 = (r-1)(r-1)(r+2)$, so

$y = c_1e^x + c_2xe^x + c_3e^{-2x}$.

12. What does it mean to say that y_1, y_2 and y_3 are independent? Show that x, $x^2 - 1$ and $x^2 - 4$ are independent using the definition.

This means that there is no non-trivial linear combination

$c_1y_1 + c_2y_2 + c_3y_3$ which is identically 0.

Suppose $c_1x + c_2(x^2-1) + c_3(x^2-4) = 0$.

For $x = 0$, this gives $-c_2 - 4c_3 = 0$.

For $x = 1$, $c_1 - 3c_3 = 0$; for $x = -1$, $-c_1 - 3c_3 = 0$.

Adding the last two equations, we see that $c_3 = 0$ and then that $c_1 = 0$ and by the first equation $c_2 = 0$ also. Thus x, $x^2 - 1$ and $x^2 - 4$ are linearly independent.

13. Find the general solution of $y^{(viii)} - y^{(vii)} - y^{(iv)} + y^{(iii)} = 0$, given that $r^8 - r^7 - r^4 + r^3 = r^3(r-1)^2(r+1)(r^2+1)$.

$y_h = c_1 + c_2x + c_3x^2 + c_4e^x + c_5xe^x + c_6e^{-x} + c_7cos\ x + c_8sin\ x$.

14. Use Undetermined Coefficients to solve $y''' - y'' + 4y' - 4y = cosx$.

The auxiliary polynomial is $r^3 - r^2 + 4r - 4r = (r-1)(r^2+4)$,

so the homogeneous solution is $y_h = c_1e^x + c_2cos\ 2x + c_3sin\ 2x$.

The particular solution is $y_p = Acos\ x + Bsin\ x$.

Then $y_p' = Bcos\ x - Asin\ x$,

$y_p'' = -Acos\ x - Bsin\ x$ and

$y_p''' = -Bcos\ x + Asin\ x$.

Thus $L[y_p] = (-3A + 3B)cos\ x + (-3A - 3B)sin\ x = cos\ x$.

So $-3A + 3B = 1$ and $-3A - 3B = 0$, which means $A = -\frac{1}{6}$ and $B = \frac{1}{6}$.

$y = y_p + y_h = -\frac{1}{6}cos\ x + \frac{1}{6}sin\ x + c_1e^x + c_2cos\ 2x + c_3sin\ 2x$.