Sample Problem Solutions for Exam Three

1. State and prove the formulas for
(a) $\mathcal{L}\{e^{at}f(t)\}$ (b) $\mathcal{L}\{f'(t)\}$
(c) $\mathcal{L}\{tf(t)\}$ (d) $\mathcal{L}\{u(t-a)f(t-a)\}$

Let $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$
(a) Let $g(t) = e^{at}f(t)$.
Then $G(s) = F(s-a)$.
Proof: $G(s) = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a)$.

(b) Let $g(t) = f'(t)$.
Then $G(s) = sF(s) - f(0)$.
Proof: $G(s) = \int_0^\infty e^{-st}f'(t)dt = [e^{-st}f(t)]_0^\infty - \int_0^\infty(-se^{-st}f(t)dt$
(using integration by parts with $u = e^{-st}$ and $dv = f'(t)dt)$
This becomes $(0 - f(0)) + s\int_0^\infty e^{-st}f(t)dt = -f(0) + sF(s)$.

(c) Let $g(t) = tf(t)$. Then $G(s) = -F'(s)$.
Proof: $F(s) = \int_0^\infty e^{-st}f(t)dt$,
so $F'(s) = \int_0^\infty \partial(e^{-st}f(t))/\partial s dt = \int_0^\infty -te^{-st}f(t)dt = -G(s)$.

(d) Let $g(t) = u(t-a)f(t-a)$.
Then $G(s) = e^{-as}F(s)$.
Proof: $G(s) = \int_0^\infty e^{-st}u(t-a)f(t-a)dt = \int_a^\infty e^{-st}f(t-a)dt = \int_0^\infty e^{-s(u+a)}f(u)du$
(substituting $u = t-a$)
This now gives $e^{-as}\int_0^\infty e^{-su}f(u)du = e^{-as}F(s)$.

2. Define the Gamma function $\Gamma(p+1)$ and show that $\mathcal{L}\{t^p\} = \Gamma(p+1)/s^{p+1}$.

$\Gamma(p+1) = \int_0^\infty x^pe^{-x}dx$.
Substituting $x = st$, so that $dx = sdt$, we get
$\Gamma(p+1) = \int_0^\infty (st)^pe^{-st}sdt = s^{p+1}\int_0^\infty e^{-st}t^pdt = s^{p+1}\mathcal{L}t^p$, so that $\mathcal{L}\{t^p\} = \Gamma(p+1)/s^{p+1}$.

3. Find the Laplace transform $F(s)$ of $f(t) =$
(a) $e^{2t}sint$ (b) $t\ cos\ 3t$ (c) $t^{5/2}$
(c) t, if $0 \le t < 1$ and t^2, if $t > 1$.

(a) Let $g(t) = sin\ t$, so that $f(t) = e^{2t}g(t)$ and $F(s) = G(s-2)$.
By the table, $G(s) = \frac{1}{s^2+1}$, so that $F(s) = \frac{1}{(s-2)^2+1} = \frac{1}{s^2-4s+5}$.

(b) Let $g(t) = cos\ 3t$, so that $f(t) = tg(t)$ and $F(s) = -G'(s)$.
By the table, $G(s) = \frac{s}{s^2+9}$, so that $F(s) = -\frac{(1)(s^2+9)-(s)(2s)}{(s^2+9)^2} = \frac{s^2-9}{(s^2+9)^2}$.

(c) $F(s) = \Gamma(7/2)/s^{7/2}$.
$\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(p+1) = p\Gamma(p)$, so
$\Gamma(7/2) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi} = \frac{15\sqrt{\pi}}{8}$ and
$F(s) = \frac{15\sqrt{\pi}}{8s^{7/2}}$.

(d) $f(t) = t+(t^2-t)u(t-1) = t+u(t-1)g(t-1)$, so that $F(s) = 1/s^2+e^{-s}G(s)$.
Now $g(t-1) = t^2-t$, so $g(t) = (t+1)^2-(t+1) = t^2+t$ and $G(s) = 2/s^3+1/s^2$.
Thus $F(s) = 1/s^2 + e^{-s}(2/s^3+1/s^2)$.

4. Find the Laplace transform $F(s)$ of the square wave $f(t)$ with period 2, where

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t < 1 \\ -1, & \text{if } 1 < t < 2 \end{cases}.$$

$f_T(t) = 1 - 2u(t-1) + u(t-2)$, so
$F_T(s) = \frac{1-2e^{-s}+e^{-2s}}{s}$, and then
$F(s) = \frac{F_T(s)}{1-e^{-2s}} = \frac{(1-e^{-s})^2}{(1-e^{-s})(1+e^{-s})s} = \frac{1-e^{-s}}{s(1+e^{-s})}$

5. Find the inverse Laplace transforms f(t) of F(s) =
(a) $(3s+8)/(s^2-8s+25)$ (b) e^{-3s}/s^4 (c) $(s+3)/(s-1)^2(s^2+4)$

(a) Completing the square, we have $s^s - 8s + 25 = (s-4)^2 + 9$.
Then $F(s) = G(s-4)$, so that $f(t) = e^{4t}g(t)$, where
$G(s) = F(s+4) = \frac{3(s+4)+8}{s^2+9} = \frac{3s+20}{s^2+9}$
so $g(t) = 3cos\ 3t + \frac{20}{3}sin\ 3t$ and $f(t) = e^{4t}(3cos\ 3t + \frac{20}{3}sin\ 3t)$.

(b) Let $G(s) = 1/s^4$ so that $F(s) = e^{-3s}G(s)$ and $f(t) = u(t-3)g(t-3)$. $g(t) = t^3/6$ so that $f(t) = u(t-3)(t-3)^3/6$.
That is, $f(t) = 0$ for $t < 3$ and $f(t) = (t-3)^3/6$ for $t > 3$.

(c) Using the method of Partial Fractions,
$\frac{s+3}{(s-1)^2(s^2+4)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+4}$.
Solving $s+3 = A(s-1)(s^2+4) + B(s^2+4) + Cs(s-1)^2 + D(s-1)^2$.
For $s = 1$, we get $4 = 5B$ so $B = 4/5$.
Equating coefficients:
$s^3 : 0 = A + C$ so $C = -A$
$s^2 : 0 = -A + B - 2C + D = A + 4/5 + D$, so $D = -A - 4/5$.
$s : 1 = 4A + C - 2D = 3A - 2D = 5A + 8/5$, so $A = -3/25$ and therefore $C = 3/25$ and $D = 3/25 - 4/5 = -17/25$.
$1 : 3 = -4A + 16/5 + D$, checking: $3 = 12/25 + 16/5 - 17/25$.
Thus $f(t) = -\frac{3}{25}e^t + \frac{4}{5}te^t + \frac{3}{25}cos\ 2t - \frac{17}{50}sin\ 2t$.

6. Solve using Laplace transforms
(a) $y'' - 3y' + 2y = 4e^{2t}$; $y(0) = 0$, $y'(0) = 1$
(b) $y'' + 25y = 10\delta(t-2)$; $y(0) = y'(0) = 0$
(c) $y'' - 4y' = 3$, if $0 < t < 2$, and $= 0$ if $2 \le t$, with $y(0) = 1$, $y'(0) = 0$.

(a) $s^2Y - 1 - 3sY + 2Y = 4/(s-2)$, so $(s^2-3s+2)Y = 1 + 4/(s-2) = (s+2)/(s-2)$ and
$Y = (s+2)/(s-1)(s-2)^2 = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$.
$s+2 = A(s-2)^2 + B(s-1)(s-2) + C(s-1)$.
$s = 1 : 3 = A$ $s = 2 : 4 = C$.
$s^2 : 0 = A + B$, so $B = -3$.
$y(t) = 3e^t - 3e^{2t} + 4te^{2t}$.

(b) $s^2Y + 25Y = 10e^{-2s}$, so $Y = 10e^{-2s}/(s^2+25) = e^{-2s}F(s)$.
Then $f(t) = 2sin\ 5t$ and $y(t) = u(t-2)f(t-2) = 2u(t-2)sin\ 5(t-2)$.
That is, $f(t) = 0$ for $t < 2$ and $f(t) = 2sin\ 5(t-2)$ for $t > 2$.

(c) The forcing function is $3 - 3u(t-2)$, which means that
$s^2Y - s - 4(sY - 1) = 3/s - 3e^{-2s}/s$, so $(s^2 - 4s)Y = s - 4 + 3/s - 3e^{-2s}/s$

Then $Y(s) = \frac{1}{s} + \frac{3}{s^2(s-4)}(1 - e^{-2s})$.
Let $F(s) = \frac{3}{s^2(s-4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-4}$.
By partial fractions $A = -\frac{3}{16}$, $B = -\frac{3}{4}$ and $C = \frac{3}{16}$.
So $f(t) = -\frac{3}{16} - \frac{3}{4}t + \frac{3}{16}e^{4t}$.
Finally $y = t + f(t) - u(t-2)f(t-2) = t - frac316 - \frac{3}{4}t + \frac{3}{16}e^{4t} - u(t-2)(-\frac{3}{16} - \frac{3}{4}t + \frac{3}{16}e^{4t-8})$
For $t < 2$, $y(t) = \frac{1}{4}t - \frac{3}{16} + \frac{3}{16}e^{4t}$ and for $t > 2$, $y(t) = t + \frac{3}{16}e^{4t} - \frac{3}{16}e^{4t-8}$

7. A mass of 4 grams on a spring with constant $k = 100$ is released from rest at time $t = 0$, 2 cm above equilibrium. Then at time $t = 3$, the mass is given an upward impulse of power 120. Write the differential equation for the position $x(t)$ of the mass at time t and use Laplace transforms to solve for $x(t)$.

$4x'' + 100x = 120\delta(t-3)$, with $x(0) = -2$ and $x'(0) = 0$.
$4s^2X + 8s + 100X = 120e^{-3s}$.
$X = \frac{-2s}{s^2+25} + e^{-3s}\frac{30}{s^2+25}$
$x = -2cos\ 5t + u(t-3)6sin\ 5(t-3)$.

8. A rocket is launched with acceleration $68 - t^2$ for time $0 \leq t \leq 10$ and acceleration -32 for $t \geq 10$. Write the differential equation for the position x(t) of the rocket at time t and use Laplace transforms to solve for x(t).

$X'' = 68 - t^2 + u(t-10)(t^2 - 100) = (68 - t^2) + u(t-10)f(t-10)$,
where $f(t-10) = t^2 - 100$ so $f(t) = (t+10)^2 - 100 = t^2 + 20t$.
Then $s^2X(s) = 68/s - 2/s^3 + e^{-10s}(2/s^3 + 20/s^2)$,
so $X(s) = 68/s^3 - 2/s^5 + e^{-10s}(2/s^5 + 20/s^4)$.
Then $x(t) = 34t^2 - t^4/12 + u(t-10)g(t-10)$, where
$G(s) = 2/s^5 + 20/s^4$ so that $g(t) = t^4/12 + 20t^3/6$.
So $x(t) = 34t^2 - t^4/12 + u(t-10)[(t-10)^4/12 + 20(t-10)^3/6]$.