1. Find the convolution $t^2 * t^3$.

$\int_0^t (t-v)^2 v^3 dv = \int_0^t (t^2 - 2vt + v^2) v^3 dv = t^2 \int_0^t v^3 dv - 2t \int_0^t v^4 dv + \int_0^t v^5 dv = \frac{1}{4}t^6 - \frac{2}{5}t^6 + \frac{1}{6}t^6 = \frac{1}{60}t^6$.

2. Use convolution to express a particular solution to $x'' + x = tan\ t$ as an integral–then evaluate.

Let $F(s) = \mathcal{L}\{tan\ t\}$.

Then $s^2X + X = F(s)$, so that $X = F(s)/(s^2+1) = F(s)G(s)$ where $g(t) = sin\ t$.

Thus $x(t) = tan\ t * sin\ t = \int_0^t tan\ u sin(t-u)du = \int_0^t tan\ u(sin\ tcos\ u - cos\ tsin\ u)du = sin\ t(\int_0^t sin\ udu) - cos\ t(\int_0^t (sec\ u - cos\ u)du) = sin\ t - cos\ tln(sec\ t + tan\ t)$.

3. Use the Taylor Series Method to find the first 4 terms of a series solution for $y' = y^2 - xy$ with $y(0) = 2$.

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

$a_0 = y(0) = 2$

$y'(0) = y(0)^2 - 0y(0) = 2^2 - 0 \cdot 2 = 4$, so $a_1 = y'(0) = 4$.

$y'' = 2yy' - y - xy'$, so $y''(0) = 2 \cdot 2 \cdot 4 - 2 = 14$ and $a_2 = y''(0)/2 = 7$.

$y''' = 2yy'' + 2(y')^2 - 2y' - xy''$, so $y'''(0) = 56 + 32 - 8 = 80$ and $a_3 = y'''(0)/6 = 80/6$.

So $y = 2 + 4x + 7x^2 + \frac{40}{3}x^3 + \cdots$.

4. Find the singular points of $(x^2-9)^2y''+(x^2-3x)y'+(x+3)y=0$ and classify them as regular or irregular.

Then find a minimum value for the radius of convergence of a power series solution about $x_0=1$.

$p(x)=\frac{x^2-3x}{(x^2-9)^2}=\frac{x}{(x-3)(x+3)^2}$

and

$q(x)=\frac{x+3}{(x^2-9)^2}=\frac{1}{(x-3)^2(x+3)}$.

The singular points are $x=3$ and $x=-3$.

For $x=3$, we have $(x-3)p=\frac{x}{x+3)^2}$ and $(x-3)^2q=\frac{1}{x+3}$. Both are analytic at $x=3$, so this is a REGULAR singular point.

For $x=-3$, we have $(x+3)p=\frac{x}{(x-3)(x+3)}$ and $(x+3)^2q=\frac{x+3}{(x-3)^2}$. The first one is not analytic at $x=-3$, so this is an IRREGULAR singular point.

The nearest singular point to $x_0=1$ is $x=3$, so the radius of convergence $R>3-1=2$.

5. Find the indicial equation of $6x^3y'''+13x^2y''+(x^2+2x)y'+xy=0$ and give the form of the general solution.

$f(r)=6r(r-1)(r-2)+13r(r-1)+2r+0=6r^3-5r^2+r=r(2r-1(3r-1)$.
The roots are $r=0$, $r=\frac{1}{2}$ and $r=\frac{1}{3}$.

The general solution is

$y=\sum_{n=0}^{\infty}a_nx^n+\sum_{n=0}^{\infty}b_nx^{n+\frac{1}{2}}+\sum_{n=0}^{\infty}c_nx^{n+\frac{1}{3}}$.

6. Find the first four terms of a power series for $\int\frac{e^x}{1-x}\,dx$.

$e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots$ and $\frac{1}{1-x}=1+x+x^2+x^3+\cdots$, so that by the Cauchy product

$\frac{e^x}{1-x}=1+2x+\frac{5}{2}x^2+\frac{8}{3}x^3+\cdots$

Then $\int\frac{e^x}{1-x}dx=x+x^2+\frac{5}{6}x^3+\frac{2}{3}x^4+\cdots$

7. Find the recurrence relation and the first 5 nonzero terms in a power series solution of $y''=2xy$ with $y(0)=6$ and $y'(0)=3$.

$y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\ldots.$

$y''=2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+\ldots$

$2xy=2a_0x+2a_1x^2+2a_2x^3+2a_3x^4+2a_4x^5+\ldots$

Equating the coefficients, we have $a_2 = 0$, $6a_3 = 2a_0$, $12a_4 = 2a_1$, $20a_5 = 2a_2$, $30a_6 = 2a_3$, and so on.

In general, $(n+3)(n+2)a_{n+3} = 2a_n$, so the recurrence formula is

$$a_{n+3} = 2a_n/(n+2)(n+3)$$

Then $a_0 = y(0) = 6$, $a_1 = y'(0) = 3$, $a_2 = 0$, $a_3 = \frac{1}{3}a_0 = 2$, $a_4 = \frac{1}{6}a_1$, $a_5 = \frac{1}{10}a_2 = 0$, $a_6 = \frac{1}{15}a_3 = \frac{2}{15}$ and so on.

Thus $y = 6 + 3x + 2x^3 + \frac{1}{2}x^4 + \frac{2}{15}x^6 + \ldots$.

8. Solve the Cauchy-Euler differential equation $x^2y'' - 5xy' + 8y = 2x^3$ with $y(1) = 3$ and $y'(1) = 5$.

The indicial equation is $r(r-1) - 5r + 8 = r^2 - 6r + 8 = (r-2)(r-4)$, so the homogeneous solution is $y_h = c_1x^2 + c_2x^4$.

Using Variation of Parameters, $y_p = v_1x^2 + v_2x^4$ and the Wronskian $W(x^2, x^4) = x^2(4x^3) - (2x)x^4 = 2x^5$.

Notice that $F(x) = 2x^3/x^2$ for Variation of Parameters.

$v_1' = \frac{-2xx^4}{2x^5} = -1$, so $v_1 = -x$.

$v_2' = \frac{2xx^2}{2x^5} = x^-2$, so $v_2 = -x^{-1}$.

Then $y_p = (-x)x^2 + (-x^{-1})x^4 = -2x^3$.

Using Undetermined Coefficients, let $y_p = Ax^3$ so that

$L[y] = x^2(6Ax) - 5x(3Ax^2) + 8(Ax^3) = -Ax^3 = 2x^3$, so that again $y_p = -2x^3$.

Now $y = y_p + y_h = c_1x^2 + c_2x^4 - 2x^3$,

so $y' = 2c_1x + 4c_2x^3 - 6x^2$. Then

$3 = y(1) = c_1 + c_2$ and $5 = y'(1) = 2c_1 + 4c_2 - 6$.

Solving $c_1 = \frac{9}{2}$ and $c_2 = \frac{1}{2}$.

$y = \frac{9}{2}x^2 + \frac{1}{2}x^4 - 2x^3$.