Calculus1

Exam Review Day

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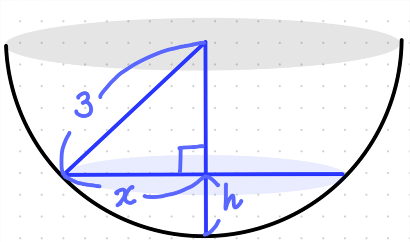
Question (15min)

Problem

Water pours into a hemispherical tank with a radius of 3 cm at a rate that increases the water height by 2 cm per second. When the water level is h cm, the volume of the hemisphere tank is given by $V = \pi(3h^2 - \frac{1}{3}h^3)$.

1. When $h = 1$, how fast does the water volume change?
2. When $h = 1$, compute $\frac{dx}{dt}$ using the following figure.



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This implies $\frac{dh}{dt} = 2$

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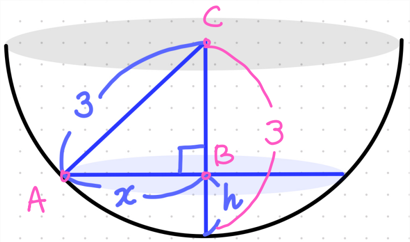


1) What we should find is the change of water volume ; $\frac{dV}{dt}$.

From the given condition, we have $\frac{dh}{dt} = 2$.

And since $V = \pi\left(3h^2 - \frac{1}{3}h^3\right)$,

(∵ Because we differentiate h by t !)

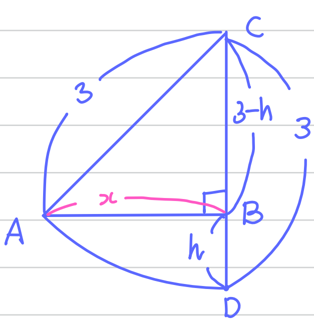
$$\frac{dV}{dt} = \pi \cdot \frac{d}{dt}\left(3h^2 - \frac{1}{3}h^3\right) = \pi \cdot (6h - h^2) \cdot \boxed{\frac{dh}{dt}}$$

$$= \pi \cdot (6h - h^2) \cdot 2 \quad \left(\because \frac{dh}{dt} = 2\right)$$

$$= \pi \cdot (12h - 2h^2)$$

∴ Here, since $h = 1$, we have $\left.\frac{dV}{dt}\right|_{h=1} = \pi \cdot (12 \cdot 1 - 2 \cdot 1^2) = \pi \cdot 10 = 10\pi$.

2) When we compute $\frac{dx}{dt}$, we need to see a triangle.



Note that a triangle ΔABC is a right triangle.

Since the given tank is a hemisphere, $\overline{CD}$ is a radius too.

Thus, since $\overline{BC} + \overline{BD} = \overline{BC} + h = 3$; $\overline{BC} = 3 - h$.

Next, use the Pythagorean theorem for ΔABC.

$$9 = 3^2 = (3-h)^2 + x^2 = (h^2 - 6h + 9) + x^2$$

$$; \ 9 - (h^2 - 6h + 9) = 6h - h^2 = x^2.$$

$$\Leftrightarrow x = \sqrt{6h - h^2} \quad (\because \text{since the length has a positive value.})$$

$\boxed{1}$ (Chain Rule)

$$\frac{dx}{dt} = \frac{d}{dt}\left(\sqrt{6h - h^2}\right) = \frac{d}{dt}\left((6h - h^2)^{1/2}\right) = \frac{1}{2} \cdot (6h - h^2)^{-1/2} \cdot (6 - 2h)\frac{dh}{dt} ;$$

When $h = 1$, we have

$$\left.\frac{dx}{dt}\right|_{h=1} = \frac{1}{2} \cdot (6 \cdot 1 - 1)^{-1/2} \cdot (6 - 2 \cdot 1) \cdot 2$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{5}} \cdot 4 \cdot 2 = \frac{4}{\sqrt{5}}.$$

$\boxed{2}$ (Implicit Differentiation)

From the equation $6h - h^2 = x^2$, (RHS) : $\frac{d}{dt}x^2 = 2x \cdot \frac{dx}{dt}$ & (LHS) : $\frac{d}{dt}(6h - h^2) = (6 - 2h)\frac{dh}{dt}$.

$\Rightarrow$ $2x \cdot \frac{dx}{dt} = (6 - 2h)\frac{dh}{dt}$; When $h = 1$, $x = \sqrt{5}$ by the Pythagorean theorem. ($3^2 = x^2 + (3-1)^2 = x^2 + 4$)

So we have ⋯ $2\sqrt{5}\,\frac{dx}{dt} = (6 - 2)\frac{dh}{dt} = 4 \cdot 2 = 8$

$$\therefore \frac{dx}{dt} = \frac{4}{\sqrt{5}} \ \blacksquare$$