Lecture 26: Representation of functions as Power Series (II)

ex. Use the geometric series (L25)

\[(\ast) \quad \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1\]

to find power series representations and its radius and interval of convergence of the following functions:
1. $f(x) = \arctan x$

The key is to remember that $\arctan x$ is an antiderivative of $1/(1 + x^2)$ which we can easily find a power series representation by substituting $(-x^2)$ for $x$ in the geometric series in (*).

Substituting $x = 0$ and noting that $\arctan 0 = 0$, the two sides of this equation agree provided we choose $C = 0$.

The power series converges for $|x| < 1$. Testing the endpoints separately, we find that:
2. \( f(x) = \frac{1}{(1 - x)^2} \)

First, we realize that \( \frac{1}{(1 - x)^2} \) is a derivative of \( \frac{1}{1 - x} \) (with possibly some coefficient) which has a power series as in (*).

We proceed to take the derivative of both sides of (*):

\[
\frac{d}{dx} \left( \frac{1}{1 - x} \right) = \frac{1}{(1 - x)^2}
\]

and the radius and interval of convergence are ____________
3. Use \( \ln(1 - t) = -\sum_{n=1}^{\infty} \frac{t^n}{n} \) over \([-1, 1)\), (derived in L25), evaluate the following indefinite integral as a power series and determine the radius and interval of convergence?

\[
\int \frac{\ln(1 - t)}{t} \, dt
\]

To start, \( \ln(1 - t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}, R = 1. \)

\[
\frac{\ln(1 - t)}{t} = \]

We now take the integral of both sides,

check end points:
NYTI: 1. Find the power series representation of the following functions:

a. \( \frac{4}{x + 2} \),  \quad b. \( \frac{x}{4 + 9x^2} \),  \quad c. \( \frac{x}{(1 - x)^3} \)

2. (a) Evaluate \( f(x) = \int \frac{1}{1 + x^5} \, dx \) as a power series.

(b) Use the first 2 terms of the power series to approximate the value of \( \int_{0}^{0.2} \frac{1}{1 + x^5} \, dx \).

3. Find a power series representations for

(a) \( \arctan(2x) \)

(b) \( \ln(5 - x) \)
4. Use the power series representation for \( \arctan x \),

(a) first approximate \( \arctan 1 \) and \( \arctan \left( \frac{1}{\sqrt{3}} \right) \),

(b) then get an approximation for \( \pi \).

(c) Why does one approximation take a fewer terms than the other to get a good approximation?
5. Find a power series and radius and interval of convergence for $f(x) = \frac{x}{(1 - 2x)^2}$

First, we note that $\frac{1}{(1 - 2x)^2}$ is a derivative of $\frac{1}{1 - 2x}$ (with possibly some coefficient).
Geometric Series Formula
\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \]

\[ R = 1, \quad IOC = (-1, 1) \]

We can find power series representation for other functions by manipulating the Geometric Series Formula via Substitution, Integration and Differentiation.

\[ \frac{1}{1+4x} \stackrel{(sub.)}{=} \sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^n, \]

Must check the Radius of convergence by checking

\[ | -4x | < 1 \rightarrow |x| < 1/4, \]

\[ R = 1/4, \quad I.O.C = (-1/4, 1/4) \]

\[ \ln(1-x) \stackrel{(int.)}{=} -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n} \]

Must check 'end points' for convergence

\[ R = 1, \quad I.O.C = [-1, 1) \]
\[
\frac{1}{(1 - x)^2} \overset{\text{diff.}}{=} - \sum_{n=1}^{\infty} nc^{n-1} = - \sum_{n=1}^{\infty} \frac{x^n}{n}
\]

\[R = 1, \quad \text{I.O.C} = (-1, 1)\]

Note: If \(\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots\)

\[\frac{d}{dx} \left( \sum_{n=0}^{\infty} x^n \right) = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}\]
On the other hand,

If \( \sum_{n=0}^{\infty} x^{2n+1} = x + x^3 + x^5 + \cdots \)

\[
\frac{d}{dx} \left( \sum_{n=0}^{\infty} x^{2n+1} \right) = 1 + 3x^2 + 5x^4 + \cdots
\]

\[
= \sum_{n=0}^{\infty} (2n + 1)x^{2n}
\]

One must be careful with the new power series index when differentiating a power series. It may be a good idea to write out a few terms of the power series, before and after differentiating, to make sure the indexing is correct.