L37 Volume of Solid of Revolution I

Disk/Washer and Shell Methods

A **solid of revolution** is a solid swept out by rotating a plane area around some straight line (the axis of revolution).

Two common methods for finding the volume of a solid of revolution are the (cross sectional) **disk method** and the (layers) of **shell method** of integration.

To apply these methods, it is easiest to:

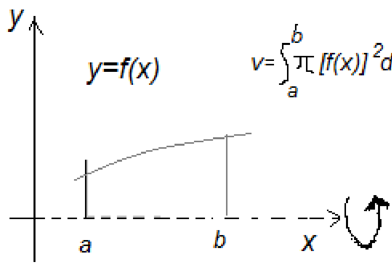
1. Draw the plane region in question;
2. Identify the area that is to be revolved about the axis of revolution;
3. Determine the volume of either a disk-shaped slice or a cylindrical shell of the solid;
4. Sum up the infinitely many disks or shells.

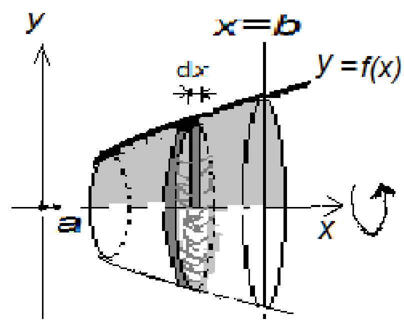
$$V = \int dV$$

Disk method

The volume V of the solid formed by rotating a plane area about the $x-$axis is given by

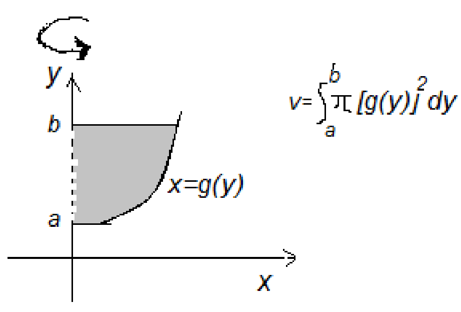
$$V = \int_a^b A(x)dx = \pi \int_a^b f^2(x)dx$$

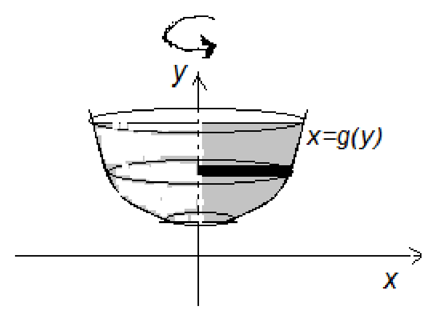




and about the $y-$axis by

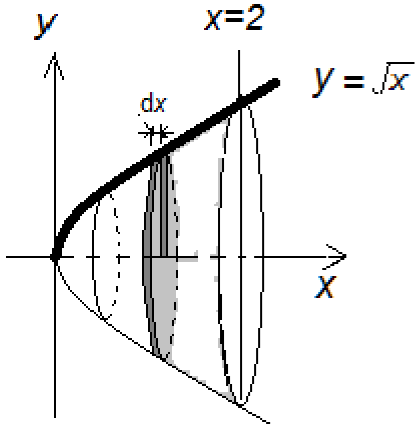
$$V = \int_a^b A(y)dy = \pi \int_a^b g^2(y)dy$$





where $A(x)$ and $A(y)$ is the cross-sectional area of the solid.

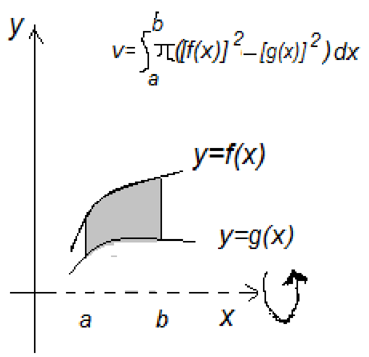
ex. Find the volume of the solid generated when the area bounded by the curve $y = \sqrt{x}$, the $x-$axis and the line $x = 2$ is revolved about the $x-$axis. $(2\pi unit^3)$

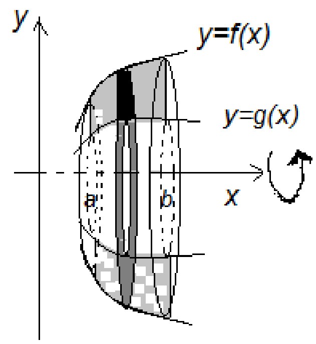


Washer Method

Alternatively, the volume of the solid formed by rotating the area between the curves of $f(x)$ (on top) and $g(x)$ (on the bottom) and the lines $x = a$ and $x = b$ about the $x-$axis is given by

$$V = \pi \int_a^b [f^2(x) - g^2(x)]dx$$





That is, we use 'washers' instead of 'disks' to obtain the volume of the 'hollowed' solid by taking the volume of the inner solid and subtract it from the volume of the outer solid.

Note:

1. $f^2 - g^2 \neq (f - g)^2$
2. To rotate about any horizontal axis, we must first calculate the outer radius (OR) and the inner radius (IR), then use the area of a washer

$$\boxed{A = \pi[(O.R.)^2 - (I.R)^2]}$$

to give us the volume of the solid of revolution

$$\boxed{V = \int_a^b \pi[(O.R.)^2 - (I.R)^2]\, dx}$$

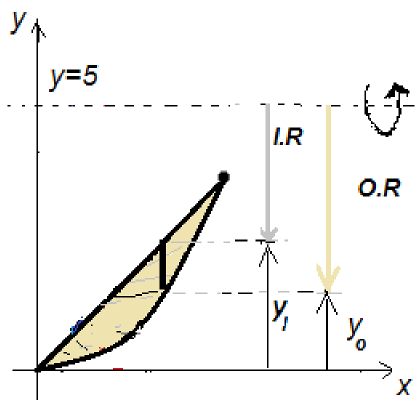
O.R.(Outer Radius) = Distance from the axis of revolution to the outer edge of the solid;

I.R.(Inner Radius) = Distance from the axis of revolution to the inner edge of the solid.

3. Same idea applies to both the $y-$axis and any other vertical axis. You simply must solve each equation for x before you plug them into the integration formula.

ex. Using the washer method, find the volume generated by rotating the region bounded by the given curves about the specified axis.

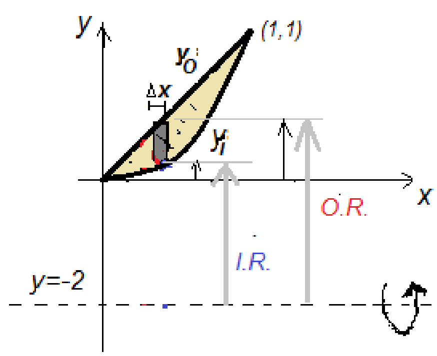
$$y = x^3,\ y = x,\ x \geq 0; \quad \text{about } y = 5. \ \left(\tfrac{97\pi}{42}\right)$$



(same as last one except about $y = -2$.)

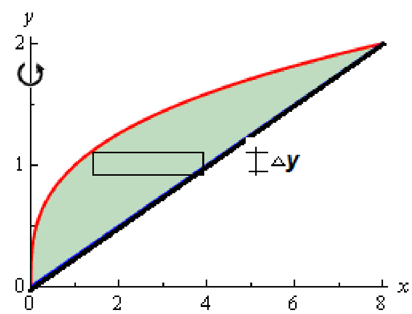
Using the washer method, find the volume generated by rotating the region bounded by the given curves about the specified axis.

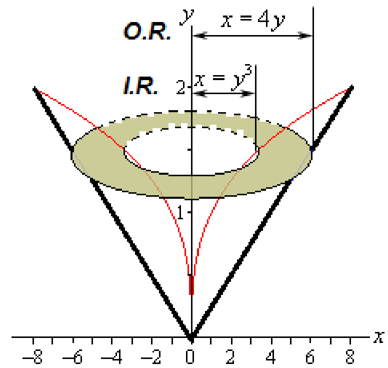
$$y = x^3,\ y = x,\ x \geq 0; \quad \text{about } y = -2.\left(\frac{25\pi}{21}\right)$$



NYTI:

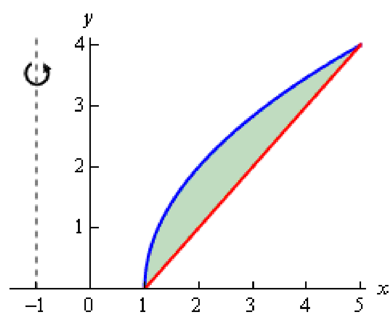
1. Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant about the $y-$axis. $(\frac{512\pi}{21})$

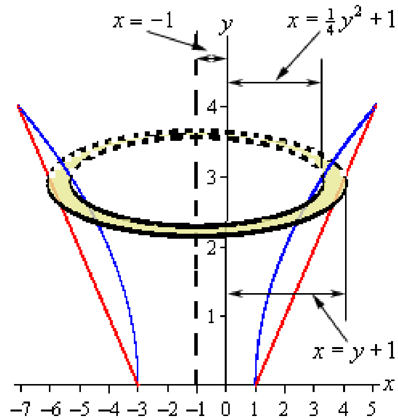




If we rotate about a horizontal axis then the cross-sectional area will be a function of x. If we rotate about a vertical axis then it will be a function of y.

2. Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = -1$. $(\frac{96\pi}{5})$





3. Using the Washer method, find the volume generated by rotating the region bounded by the given curves about the specified axis.

$$y = (x-1)^{1/2},\ y = 0,\ x = 5;\ \text{about } y = 3.$$

(24π)