Lecture 7: Techniques of Integration

IV. Integration of Rational Functions by Partial Fractions, part I (7.5)

Which integral would you rather evaluate?

\[
\int \left( \frac{2}{x - 1} - \frac{1}{x + 2} \right) \, dx \quad \text{or} \quad \int \frac{x + 5}{x^2 + x - 2} \, dx
\]

The method of Partial Fractions allows us to integrate a rational function by first expressing it as a sum of simpler fractions called Partial Fractions that can be integrated easily.
A rational function \( f(x) = \frac{P(x)}{Q(x)} \) is \textbf{proper} if \( \deg(P) < \deg(Q) \).

If \( f(x) \) is \textbf{improper}, we use \textbf{long division} to write

\[
f(x) = \frac{P(x)}{Q(x)} = g(x) + \frac{R(x)}{Q(x)}, \quad \deg(R) < \deg(Q).
\]

where \( g, P, R, Q \) are polynomials.

\textbf{ex.} Write \( \frac{x^3 + x}{x - 1} \) as a proper rational expression.
Factor the denominator $Q(x)$ as far as possible:

**Fact:** Any polynomials $Q(x)$ of real coefficients can be factored as a product of linear and/or irreducible quadratic factors. This gives us 4 possible cases of decomposing a proper rational function. We start with the simplest case:

**Case 1** $Q(x)$ is a product of distinct linear factors with no repeats.

$$Q(x) = (a_1x + b_1) \cdots (a_nx + b_n)$$

then, there is a partial fraction decomposition (PFD)

$$f(x) = \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \cdots + \frac{A_n}{a_nx + b_n}$$

for suitable constants $A_1, \cdots A_n$.

Once we have found the PFD, we can integrate the rational function $f(x)$ more easily.
**ex.** First, practicing writing out the **form** of the partial fraction **decomposition**.
(Do not determine the constants.)

1. \[
\frac{3x}{(2x + 3)(x - 1)}
\]

2. \[
\frac{x^2 + 1}{x^2 - 1} \quad (1 + \frac{A}{x+1} + \frac{B}{x-1})
\]
Evaluating the following integrals:

**ex.** \[ \int \frac{1}{(x + 4)(x - 1)} \, dx. \]

**PFD:**

\[
\frac{1}{(x + 4)(x - 1)} = \frac{x + 4}{x - 1} + \frac{1}{x - 1} \quad (\star)
\]

**Finding the Constants:**

1. Multiply both sides by \((x + 4)(x - 1)\) to clear denominators:
   
   \[ 1 = A(x - 1) + B(x + 4) \]

2. Let \(x = -4\) (to make the \(B\)’s term disappear)
   
   \[ 1 = -5A \implies A = -\frac{1}{5} \]

3. Let \(x = 1\) (to make the \(A\)’s term disappear)
   
   \[ 1 = 5B \implies B = \frac{1}{5} \]

The resulting PFD is

\[
\frac{1}{(x + 4)(x - 1)} = \frac{1}{5} \left( \frac{1}{x - 1} - \frac{1}{x + 4} \right)
\]
Evaluate:

\[ \int \frac{1}{(x+4)(x-1)} \, dx = \frac{1}{5} \int \left( \frac{1}{x-1} - \frac{1}{x+4} \right) \, dx. \]

\[ = \frac{1}{5} \left( \ln |x-1| - \ln |x+4| \right) \]

\[ = \frac{1}{5} \ln \left| \frac{x-1}{x+4} \right| + c \]

There is a quick way to determine these constants (⋆) by using the

**Cover-Up Method**:

To find A:

Cover B’s term on RHS, and \((x+4)\) on both sides, plug in \(x = -4 \rightarrow A = -\frac{1}{5} \).

To find B:

Cover A’s term on RHS, and \((x-1)\) on both sides, plug in \(x = 1 \rightarrow B = \frac{1}{5} \).
Recap the 4 steps in integrating a rational expression using partial fractions—

1. First, make sure the expression is **Proper**
2. **Partial Fraction Decomposition** (PFD)
3. Finding the **Constants**
4. Integrate the **Partial Fractions**.

**ex.** \[ \int_1^2 \frac{4y^2 - 7y - 12}{y(y + 2)(y - 3)} \, dy \]

**PFD:** \[ \frac{4y^2 - 7y - 12}{y(y + 2)(y - 3)} = \]

**Constants:** (use cover-up method:)

Let \( y = 0 \), \( A = \)

Let \( y = -2 \), \( B = \)

Let \( y = 3 \), \( C = \)
Evaluate:

\[ \int_1^2 \frac{4y^2 - 7y - 12}{y(y + 2)(y - 3)} \, dy = \int_1^2 \left( \frac{-7}{y + 2} + \frac{9}{y - 3} \right) \, dy \]

\[ = \frac{9}{5} \ln(\frac{8}{3}). \]
Case 2 $Q(x)$ has repeated linear factors

$$Q(x) = (ax + b)^n,$$

then, there is a PFD

$$\frac{R(x)}{Q(x)} = \frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n}$$

for suitable constants $A_1, \cdots A_n$.

Practicing writing out the form of the partial fraction decomposition for the case 2 type.
(Do not determine the constants.)

3. $\frac{x^2 + 9x - 12}{(3x - 1)(x + 6)^2}$

4. $\frac{1}{x^4 - x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} + \frac{D}{x - 1}$
\[
\int \frac{1}{(x + 5)^2(x - 1)} \, dx.
\]

**PFD:**
\[
\frac{1}{(x + 5)^2(x - 1)} =
\]

**Constants:**

**Cover-Up:** \( x = 1, \)

\( x = -5, \)

3rd constant **can’t** be found the same way, but...

Plug in \( x = \)

**Evaluate:**
\[
\int \frac{1}{(x + 5)^2(x - 1)} \, dx = \int \frac{1}{36 \ln |\frac{x-1}{x+5}|} + \frac{1}{6(x+5)} + c
\]
NYTI: \[ \int \frac{ds}{s^2(s - 1)^2} \]

PFD: \[ \frac{1}{s^2(s - 1)^2} = \]

Constants:

Cover-Up: \[ s = 0, \]
\[ s = 1, \]

To find the other coefficients:

Evaluate:

\[ \int \frac{ds}{s^2(s - 1)^2} = \int ds. \]

\[ = 2 \ln \left| \frac{s}{s-1} \right| - \frac{1}{s} - \frac{1}{s-1} + c \]