Section 1.1: Background

Motivating Examples

Definition. A differential equation (sometimes abbreviated DE) is an equation containing derivatives of an unknown function.

Differential equations often arise when trying to explain physical phenomena by using mathematical models. One familiar example from physics and introductory calculus is *free fall*. In this scenario, we release an object from a certain height and allow it to fall under the force of gravity (we ignore other forces, such as air resistance). By Newton's second law of motion, $\mathbf{F} = ma$, we obtain the differential equation

$$m\frac{d^2h}{dt^2} = -mg,$$

where m is the mass of the object, $h(t)$ is the height above the ground at time t, and g is the gravitational constant. This particular DE can be solved simply by integration:

$$\frac{d^2h}{dt^2} = -g \Rightarrow \frac{dh}{dt} = -gt + C_1 \Rightarrow h(t) = \frac{-gt^2}{2} + C_1t + C_2.$$

The constants C_1, C_2 can be determined if we know the initial velocity and initial height of the object.

Another familiar example is radioactive decay. Assuming that the rate of decay is proportional to the amount of radioactive substance present, we obtain the DE

$$\frac{dA}{dt} = -kA,$$

where $A(t)$ is the amount of radioactive substance present at time t and $k > 0$ is the proportionality constant. Again, this DE can be solved by integration:

$$\frac{1}{A}dA = -k\ dt \Rightarrow \ln A = -kt + C_1 \Rightarrow A(t) = e^{-kt}e^{C_1} = Ce^{-kt}.$$

Again, the constant C can be determined if we know the initial amount of radioactive substance.

Let us make two fundamental observations: the solution of a DE is a *function*, NOT a number. Also, we expect that a DE *will not have a unique solution*, since there are constants of integration involved.

Of course, we cannot expect all DE's to be as simple as the ones we have introduced above. Here are a few additional examples.

Example 1. If $P(t)$ represents the value of a savings account after t years which pays a yearly interest rate of $r\%$ compounded continuously, then P satisfies

$$\frac{dP}{dt} = \frac{r}{100}P$$

Example 2. Suppose we have an electric circuit with a resistor of resistance R, an inductor with inductance L, and a capacitor of capacitance C and charge $q(t)$ which is driven by an electromotive force $E(t)$. Then applying Kirchhoff's laws yields

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Example 3. In wave propagation of vibrating strings, if x is the location along the string, c is the wave speed, and $u(x,t)$ is the displacement of the string, then

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Foundational Definitions

Let us now introduce the foundational terminology. In an equation involving a derivative, the variable we are taking the derivative of is called a **dependent variable** (this will be in the numerator of the differential), while the variable we are taking the derivative with respect to is the **independent variable** (this will be in the denominator of the differential). For example, in the differential $\frac{dy}{dx}$, y is the dependent variable and x is the independent variable.

One classification of DE's corresponds to the type of derivative involved. If a DE contains only ordinary derivatives with respect to a single independent variable, we call it an **ordinary differential equation**. However, if a DE contains partial derivatives with respect to multiple independent variables, we call it a **partial differential equation**. This course will focus entirely on ordinary DE's.

Definition. The order of a differential equation is the value of the highest-order derivative present in the equation.

Another classification of ODE's looks at the powers of the derivatives involved.

Definition. A differential equation is linear if it has the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_o(x)y = F(x).$$

Otherwise, we say the differential equation is nonlinear.

This definition seems intimidating at first glance, but here are the important points: (1) the dependent variable (here, y) and its derivatives are raised only to the first power (hence the term "linear"); (2) the coefficients of these terms, and the right hand side, should depend *only* on the independent variable (here, x). This distinction of linear/nonlinear is useful because linear ODE's are easier to solve than nonlinear ones, much as tangent lines help us to understand higher dimensional curves. In practice, most DE's you will encounter will be nonlinear.

To practice these terms, let us find the relevant information for the previously introduced DE's.

- The free fall DE is a second-order ordinary DE which is linear.
- The radioactive decay DE is a first-order ordinary DE which is also linear.
- The DE in Example 1 is also a linear first-order ordinary DE.
- The DE in Example 2 is a linear second-order ordinary DE.
- The DE in Example 3 is a second-order partial DE.

Here are a couple examples of nonlinear DE's:

Example 4. $\frac{d^2y}{dx^2} + y^3 = 0$ is nonlinear because the y term is raised to a power larger than 1.

Example 5. $\frac{d^2y}{dx^2} - y\frac{dy}{dx} = \cos x$ is nonlinear because the coefficient of dy/dx depends on the dependent variable y.

Homework: p. 5 #1-15 odd