Section 1.2: Solutions and Initial Value Problems

Explicit Solutions

When solving a DE, the best possible result is that we are able to obtain an explicit solution.

Definition. If a function $\phi(x)$ is substituted for y in a DE and satisfies the equation for all x in some interval I, we say it is an explicit solution to the equation on I.

Example 1. Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the linear equation $\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0$, but $\psi(x) = x^3$ is not.

Solution. We calculate $\phi'(x) = 2x + x^{-2}, \phi''(x) = 2 - 2x^{-3}$ and observe these functions are defined for all $x \neq 0$. Substitution into the DE yields

$$(2 - 2x^{-3}) - \frac{2}{x^2}(x^2 - x^{-1}) = (2 - 2x^{-3}) - (2 - 2x^{-3}) = 0.$$

Therefore, $\phi(x)$ is an explicit solution to the DE on $(-\infty, 0)$ and on $(0, \infty)$.

For $\psi(x) = x^3$, we have $\psi'(x) = 3x^2, \psi''(x) = 6x$ which are defined everywhere; substitution into the DE yields

$$6x - \frac{2}{x^2}x^3 = 4x = 0,$$

but this is valid only at the point $x = 0$ and not on an interval. Therefore, $\psi(x)$ is not a solution. $\Diamond$

Example 2. Show that for any constants c_1 and c_2, the function $\phi(x) = c_1e^{-x} + c_2e^{2x}$ is an explicit solution to the linear equation $y'' - y' - 2y = 0$.

Solution. We have $\phi'(x) = -c_1e^{-x} + 2c_2e^{2x}$, $\phi''(x) = c_1e^{-x} + 4c_2e^{2x}$. Substitution into the DE yields

$$\begin{aligned}&(c_1e^{-x} + 4c_2e^{2x}) - (-c_1e^{-x} + 2c_2e^{2x}) - 2(c_1e^{-x} + c_2e^{2x})\\ &= (c_1 + c_1 - 2c_1)e^{-x} + (4c_2 - 2c_2 - 2c_2)e^{2x} = 0.\end{aligned}$$

Since the functions are all defined everywhere, $\phi(x)$ is an explicit solution to the DE on $(-\infty, \infty)$ for any choice of the constants c_1, c_2. $\Diamond$

Implicit Solutions

In many cases, we will not be able to find an explicit function which satisfies a DE, but we will instead find an implicit function, which is a relation between the dependent and independent variables that cannot be solved for the dependent variable.

Definition. We say that a relation $G(x, y) = 0$ is an implicit solution to a DE on the interval I if it defines at least one explicit solution on I.

Technically, when given a relation $G(x, y) = 0$ we need something called the *implicit function theorem* in order to determine that the relation actually defines a function $y(x)$ - this conclusion is not automatic. However, in this section we will assume that the implicit function theorem applies, and use implicit differentiation to verify solutions of DE's.

Example 3. Show that $x + y + e^{xy} = 0$ is an implicit solution to the nonlinear equation $(1 + xe^{xy})\dfrac{dy}{dx} + 1 + ye^{xy} = 0$.

Solution. By implicit differentiation of the relation, we have

$$\frac{d}{dx}(x + y + e^{xy}) = 1 + \frac{dy}{dx} + e^{xy}\left(y + x\frac{dy}{dx}\right) = 0.$$

Rearranging terms gives

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0,$$

which is exactly the DE we wanted to satisfy. ♢

Example 4. Verify that for every constant C, the relation $4x^2 - y^2 = C$ is an implicit solution to $y\dfrac{dy}{dx} - 4x = 0$. Graph the solution curves for $C = 0, \pm 1, \pm 4$.

Solution. Again, we implicitly differentiate the relation to get

$$8x - 2y\frac{dy}{dx} = 0 \Leftrightarrow -2(y\frac{dy}{dx} - 4x) = 0 \Leftrightarrow y\frac{dy}{dx} - 4x = 0,$$

as desired. The solution curve for $C = 0$ is $y^2 = 4x^2 \Rightarrow y = \pm 2x$, a pair of lines passing through the origin. If $C = \pm 1, \pm 4$, or any other nonzero value, the solution curve is a hyperbola with $y = \pm 2x$ as asymptotes. ♢

Remark. If as in this example we get a solution which involves a single constant C, we call the collection of solution curves for all possible values of C a *one-parameter family of solutions*. More generally, if the solution involves n constants, the collection of solution curves is an *n-parameter family of solutions*.

Initial Value Problems

Recall in Section 1.1 that the solution of the first-order radioactive decay DE involved a single constant ($A(t) = Ce^{-kt}$), while the second-order free fall DE had two constants of integration ($h(t) = \dfrac{-gt^2}{2} + c_1 t + c_2$). One might intuitively guess that the solution to an order n DE will involve n arbitrary constants, and we will show later in the course that this is correct. These constants can be determined if we are given initial values for each of the lower-order derivatives: $y(x_0), y'(x_0), ..., y^{(n-1)}(x_0)$. When these values are specified, we can find a *particular solution* rather than a *general solution*, and problems of this type are called initial value problems (often abbreviated IVP).

Example 5. Show that $\phi(x) = \sin x - \cos x$ is a solution to the IVP

$$\frac{d^2y}{dx^2} + y = 0; y(0) = -1, y'(0) = 1.$$

Solution. We find $\phi'(x) = \cos x + \sin x, \phi''(x) = -\sin x + \cos x$, which are both defined everywhere. Substituting into the DE gives

$$(-\sin x + \cos x) + (\sin x - \cos x) = 0,$$

so the DE is satisfied on the interval $(-\infty, \infty)$. Checking the initial conditions, we have

$$\phi(0) = \sin 0 - \cos 0 = -1, \phi'(0) = \cos 0 + \sin 0 = 1,$$

as desired. Therefore, $\phi(x)$ is a solution to the IVP. $\Diamond$

Example 6. For the DE in Example 2, determine the constants c_1, c_2 so that the initial conditions $y(0) = 2$ and $y'(0) = -3$ are satisfied.

Solution. Recall that the DE had solution $\phi(x) = c_1e^{-x} + c_2e^{2x} \Rightarrow \phi'(x) = -c_1e^{-x} + 2c_2e^{2x}$. Substituting the initial conditions gives a system of equations:

$$\begin{array}{rcrcr} c_1 & + & c_2 & = & 2 \\ -c_1 & + & 2c_2 & = & -3 \end{array}$$

Adding the equations gives $3c_2 = -1 \Rightarrow c_2 = -1/3$. Substituting this into the first equation, we have $c_1 = 2 - c_2 = \frac{6}{3} - (-\frac{1}{3}) = \frac{7}{3}$. So a solution to the IVP is

$$\phi(x) = \frac{7}{3}e^{-x} - \frac{1}{3}e^{2x}. \qquad \Diamond$$

Homework: pp. 13-15 #1-11 odd, 16, 17, 22