Section 1.2: Solutions and Initial Value Problems

**Explicit Solutions**

When solving a DE, the best possible result is that we are able to obtain an explicit solution.

**Definition.** If a function \( \phi(x) \) is substituted for \( y \) in a DE and satisfies the equation for all \( x \) in some interval \( I \), we say it is an explicit solution to the equation on \( I \).

**Example 1.** Show that \( \phi(x) = x^2 - x^{-1} \) is an explicit solution to the linear equation \( \frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0 \), but \( \psi(x) = x^3 \) is not.

**Solution.** We calculate \( \phi'(x) = 2x + x^{-2}, \phi''(x) = 2 - 2x^{-3} \) and observe these functions are defined for all \( x \neq 0 \). Substitution into the DE yields

\[
(2 - 2x^{-3}) - \frac{2}{x^2}(x^2 - x^{-1}) = (2 - 2x^{-3}) - (2 - 2x^{-3}) = 0.
\]

Therefore, \( \phi(x) \) is an explicit solution to the DE on \((-\infty, 0) \) and on \((0, \infty) \).

For \( \psi(x) = x^3 \), we have \( \psi'(x) = 3x^2, \psi''(x) = 6x \) which are defined everywhere; substitution into the DE yields

\[
6x - \frac{2}{x^2}x^3 = 4x = 0,
\]

but this is valid only at the point \( x = 0 \) and not on an interval. Therefore, \( \psi(x) \) is not a solution.

**Example 2.** Show that for any constants \( c_1 \) and \( c_2 \), the function \( \phi(x) = c_1 e^{-x} + c_2 e^{2x} \) is an explicit solution to the linear equation \( y'' - y' - 2y = 0 \).

**Solution.** We have \( \phi'(x) = -c_1 e^{-x} + 2c_2 e^{2x}, \phi''(x) = c_1 e^{-x} + 4c_2 e^{2x} \). Substitution into the DE yields

\[
(c_1 e^{-x} + 4c_2 e^{2x}) - (-c_1 e^{-x} + 2c_2 e^{2x}) - 2(c_1 e^{-x} + c_2 e^{2x}) = (c_1 + c_1 - 2c_1)e^{-x} + (4c_2 - 2c_2 - 2c_2)e^{2x} = 0.
\]

Since the functions are all defined everywhere, \( \phi(x) \) is an explicit solution to the DE on \((-\infty, \infty) \) for any choice of the constants \( c_1, c_2 \).

**Implicit Solutions**

In many cases, we will not be able to find an explicit function which satisfies a DE, but we will instead find an implicit function, which is a relation between the dependent and independent variables that cannot be solved for the dependent variable.

**Definition.** We say that a relation \( G(x, y) = 0 \) is an implicit solution to a DE on the interval \( I \) if it defines at least one explicit solution on \( I \).
Technically, when given a relation \( G(x, y) = 0 \) we need something called the *implicit function theorem* in order to determine that the relation actually defines a function \( y(x) \) — this conclusion is not automatic. However, in this section we will assume that the implicit function theorem applies, and use implicit differentiation to verify solutions of DE’s.

**Example 3.** Show that \( x + y + e^{xy} = 0 \) is an implicit solution to the nonlinear equation \( (1 + xe^{xy}) \frac{dy}{dx} + 1 + ye^{xy} = 0 \).

**Solution.** By implicit differentiation of the relation, we have
\[
\frac{d}{dx}(x + y + e^{xy}) = 1 + \frac{dy}{dx} + e^{xy} \left( y + x \frac{dy}{dx} \right) = 0.
\]
Rearranging terms gives
\[
(1 + xe^{xy}) \frac{dy}{dx} + 1 + ye^{xy} = 0,
\]
which is exactly the DE we wanted to satisfy.

**Example 4.** Verify that for every constant \( C \), the relation \( 4x^2 - y^2 = C \) is an implicit solution to \( y \frac{dy}{dx} - 4x = 0 \). Graph the solution curves for \( C = 0, \pm 1, \pm 4 \).

**Solution.** Again, we implicitly differentiate the relation to get
\[
8x - 2y \frac{dy}{dx} = 0 \iff -2(y \frac{dy}{dx} - 4x) = 0 \iff y \frac{dy}{dx} - 4x = 0,
\]
as desired. The solution curve for \( C = 0 \) is \( y^2 = 4x^2 \Rightarrow y = \pm 2x \), a pair of lines passing through the origin. If \( C = \pm 1, \pm 4 \), or any other nonzero value, the solution curve is a hyperbola with \( y = \pm 2x \) as asymptotes.

**Remark.** If as in this example we get a solution which involves a single constant \( C \), we call the collection of solution curves for all possible values of \( C \) a *one-parameter family of solutions*. More generally, if the solution involves \( n \) constants, the collection of solution curves is an *\( n \)-parameter family of solutions*.

**Initial Value Problems**

Recall in Section 1.1 that the solution of the first-order radioactive decay DE involved a single constant \( (A(t) = Ce^{-kt}) \), while the second-order free fall DE had two constants of integration \( (h(t) = \frac{-gt^2}{2} + c_1 t + c_2) \). One might intuitively guess that the solution to an order \( n \) DE will involve \( n \) arbitrary constants, and we will show later in the course that this is correct. These constants can be determined if we are given initial values for each of the lower-order derivatives: \( y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0) \). When these values are specified, we can find a *particular solution* rather than a *general solution*, and problems of this type are called *initial value problems* (often abbreviated IVP).
Example 5. Show that $\phi(x) = \sin x - \cos x$ is a solution to the IVP

$$\frac{d^2y}{dx^2} + y = 0; y(0) = -1, y'(0) = 1.$$  

Solution. We find $\phi'(x) = \cos x + \sin x, \phi''(x) = -\sin x + \cos x$, which are both defined everywhere. Substituting into the DE gives

$$(\sin x - \cos x) + (\sin x - \cos x) = 0,$$

so the DE is satisfied on the interval $(-\infty, \infty)$. Checking the initial conditions, we have

$$\phi(0) = \sin 0 - \cos 0 = -1, \phi'(0) = \cos 0 + \sin 0 = 1,$$

as desired. Therefore, $\phi(x)$ is a solution to the IVP.  

Example 6. For the DE in Example 2, determine the constants $c_1, c_2$ so that the initial conditions $y(0) = 2$ and $y'(0) = -3$ are satisfied.

Solution. Recall that the DE had solution $\phi(x) = c_1 e^{-x} + c_2 e^{2x} \Rightarrow \phi'(x) = -c_1 e^{-x} + 2c_2 e^{2x}$. Substituting the initial conditions gives a system of equations:

$$c_1 + c_2 = 2$$

$$-c_1 + 2c_2 = -3$$

Adding the equations gives $3c_2 = -1 \Rightarrow c_2 = -1/3$. Substituting this into the first equation, we have $c_1 = 2 - c_2 = \frac{6}{3} - (-\frac{1}{3}) = \frac{7}{3}$. So a solution to the IVP is

$$\phi(x) = \frac{7}{3} e^{-x} - \frac{1}{3} e^{2x}.$$  

Homework: pp. 13-15 #1-11 odd, 16, 17, 22