Section 1.3: Direction Fields

Direction Fields

We want to get as much information as we can about the solution of a DE: its value at a certain point, intervals where the solution is increasing/decreasing, where the solution obtains a maximum/minimum, etc. Of course, these things are easily accomplished if we have an explicit solution, but in most real-world applications, we will not be able to find such a solution, so we need to develop other techniques and tools for obtaining the desired information about the solution. A useful strategy for graphing the solutions to a first-order DE is sketching the direction field of the equation.

Writing the first-order DE in the form \( \frac{dy}{dx} = f(x, y) \), we see that the function \( f(x, y) \) specifies the slope of a solution curve to the DE at any point \((x, y)\) in the domain of \( f \). Much like vector fields in multivariable calculus, for every point \((x_0, y_0)\) in the domain of \( f \), we can draw a short line segment whose slope is the value \( f(x_0, y_0) \); the graph of all these line segments is called the direction field (or slope field). This field describes the “flow” of solution curves, so it is easy to picture the general shape of a particular solution. The direction field also allows us to determine asymptotic behavior (limiting values as \( x \to \pm \infty \)). However, the method does have its limitations: it is difficult to unambiguously determine a particular solution curve.

The slopes are usually calculated just by plugging in values; this is easiest when the DE has the form \( \frac{dy}{dx} = f(y) \) because the right-hand side depends only on the dependent variable, and not the independent variable. This means that along every horizontal line \( y = b \), the slopes are all the same. Equations of this type are called autonomous.

Example 1. The logistic equation for the population \( p \) in thousands of a species at time \( t \) is

\[
\frac{dp}{dt} = p(2 - p).
\]

After sketching the direction field, answer the following questions:
(a) If the initial population is 3000, what will be the limiting population \( \lim_{t \to +\infty} p(t) \)?
(b) Can a population of 1000 ever decline to 500?
(c) Can a population of 1000 ever increase to 3000?

Solution. Sketch the direction field by plugging in various values for \( p \); since the DE is autonomous, this determines the slope for all points on the line \( p = b \). We have \( y'(0) = 0, y'(1) = 1, y'(2) = 0, y'(3) = -3 \), etc.
(a) All nonzero solution curves approach the line \( p = 2 \) as \( t \to +\infty \).
(b) No; populations of 1000 will increase, approaching 2000.
(c) No, it will increase but not to more than 2000.
**Isoclines**

If the DE is not autonomous, but the function $f(x, y)$ is not too complicated, we can use the method of isoclines described below to sketch the direction field.

**Definition.** An isocline for the DE $y' = f(x, y)$ is a set of points in the $xy$-plane where all solutions have the same slope $y'$; in other words, it is a level curve for the function $f(x, y)$.

Finding the isoclines of a DE, then, reduces to finding an equation for the level curve $f(x, y) = C$ for various values of $C$. Along each such curve, we can draw line segments with slope $C$ to produce the direction field.

**Example 2.** Find isoclines of the DE $y' = x + y$ for $C = 0, \pm 1, \pm 2$ and plot the corresponding direction field.

**Solution.** In general, $x + y = C \Rightarrow y = -x + C$, so the isoclines (level curves) are lines of slope $-1$ with $y$-intercept $C$.

**Example 3.** Find isoclines of the DE $y' = x^2 - y$ for $C = 0, \pm 1, \pm 2$ and plot the corresponding direction field.

**Solution.** In general, $x^2 - y = C \Rightarrow y = x^2 - C$, so the isoclines are upward parabolas with $y$-intercept $-C$.

Homework: pp. 21-22 #3, 5-8, 11-15 odd.