Section 1.3: Direction Fields

<u>Direction Fields</u>

We want to get as much information as we can about the solution of a DE: its value at a certain point, intervals where the solution is increasing/decreasing, where the solution obtains a maximum/minimum, etc. Of course, these things are easily accomplished if we have an explicit solution, but in most real-world applications, we will not be able to find such a solution, so we need to develop other techniques and tools for obtaining the desired information about the solution. A useful strategy for graphing the solutions to a first-order DE is sketching the direction field of the equation.

Writing the first-order DE in the form $\dfrac{dy}{dx} = f(x, y)$, we see that the function $f(x, y)$ specifies the slope of a solution curve to the DE at any point (x, y) in the domain of f. Much like vector fields in multivariable calculus, for every point (x_0, y_0) in the domain of f, we can draw a short line segment whose slope is the value $f(x_0, y_0)$; the graph of all these line segments is called the *direction field* (or slope field). This field describes the "flow" of solution curves, so it is easy to picture the general shape of a particular solution. The direction field also allows us to determine asymptotic behavior (limiting values as $x \to \pm\infty$). However, the method does have its limitations: it is difficult to unambiguously determine a particular solution curve.

The slopes are usually calculated just by plugging in values; this is easiest when the DE has the form $\dfrac{dy}{dx} = f(y)$ because the right-hand side depends only on the dependent variable, and not the independent variable. This means that along every horizontal line $y = b$, the slopes are all the same. Equations of this type are called **autonomous**.

Example 1. The logistic equation for the population p in thousands of a species at time t is

$$\frac{dp}{dt} = p(2 - p).$$

After sketching the direction field, answer the following questions:
(a) If the initial population is 3000, what will be the limiting population $\lim_{t\to+\infty} p(t)$?
(b) Can a population of 1000 ever decline to 500?
(c) Can a population of 1000 ever increase to 3000?

Solution. Sketch the direction field by plugging in various values for p; since the DE is autonomous, this determines the slope for all points on the line $p = b$. We have $y'(0) = 0, y'(1) = 1, y'(2) = 0, y'(3) = -3$, etc.
(a) All nonzero solution curves approach the line $p = 2$ as $t \to +\infty$.
(b) No; populations of 1000 will increase, approaching 2000.
(c) No, it will increase but not to more than 2000. $\Diamond$

Isoclines

If the DE is not autonomous, but the function $f(x, y)$ is not too complicated, we can use the method of isoclines described below to sketch the direction field.

Definition. An isocline for the DE $y' = f(x, y)$ is a set of points in the xy-plane where all solutions have the same slope y'; in other words, it is a level curve for the function $f(x, y)$.

Finding the isoclines of a DE, then, reduces to finding an equation for the level curve $f(x, y) = C$ for various values of C. Along each such curve, we can draw line segments with slope C to produce the direction field.

Example 2. Find isoclines of the DE $y' = x + y$ for $C = 0, \pm 1, \pm 2$ and plot the corresponding direction field.

Solution. In general, $x + y = C \Rightarrow y = -x + C$, so the isoclines (level curves) are lines of slope -1 with y-intercept C. $\Diamond$

Example 3. Find isoclines of the DE $y' = x^2 - y$ for $C = 0, \pm 1, \pm 2$ and plot the corresponding direction field.

Solution. In general, $x^2 - y = C \Rightarrow y = x^2 - C$, so the isoclines are upward parabolas with y-intercept $-C$. $\Diamond$

Homework: pp. 21-22 #3, 5-8, 11-15 odd.