

Section 1.4: The Approximation Method of Euler

There are three general approaches to solving DE's. The analytic approach will be the major focus of the course. We were introduced to a graphical approach in Section 1.3, where we got information about solutions by sketching direction fields. In this section, we study an example of a numerical approach. When analytic techniques fail, numerical methods can produce an approximation that is "good enough", provided that one understands how to make the method as accurate as possible.

Consider the IVP $y' = f(x, y)$, $y(x_0) = y_0$. The idea is to start at the given initial point (x_0, y_0) , and follow the tangent line to the solution curve to the next point (x_1, y_1) , then follow the tangent line at *this* point to the next point (x_2, y_2) , etc. until we arrive at the point whose value we are trying to approximate. To be precise, fix a small positive number h as the step size, and let the x -values be equally spaced, given by the formula

$$x_n = x_0 + nh, n = 0, 1, 2, \dots$$

Since the slope of the solution curve $\phi(x)$ is given by $f(x, y)$, the tangent line to the solution at the initial point (x_0, y_0) is

$$y = y_0 + (x - x_0)f(x_0, y_0).$$

Therefore, at the point $x_1 = x_0 + h$ we have

$$y_1 = y_0 + hf(x_0, y_0),$$

which is an approximation for $\phi(x_1)$. Likewise, the tangent line to $\phi(x)$ at the new point is

$$y = y_1 + (x - x_1)f(x_1, y_1),$$

so the next approximation is

$$\phi(x_2) \approx y_2 = y_1 + hf(x_1, y_1).$$

In summary, given the initial condition of an IVP, we can approximate values of the solution at equally spaced intervals according to the following recursive formulas:

$$\begin{cases} x_{n+1} = x_n + h, \\ y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, 2, \dots \end{cases}$$

Example 1. Use Euler's method with step size $h = 0.1$ to approximate the solution to the IVP

$$y' = x\sqrt{y}, \quad y(1) = 4$$

at the points $x = 1.1, 1.2, 1.3, 1.4, 1.5$.

Solution. We have $x_0 = 1, y_0 = 4, h = 0.1$, and $f(x, y) = x\sqrt{y}$. Then $y_{n+1} = y_n + 0.1x_n\sqrt{y_n}$. We calculate

$$\begin{aligned} y_1 &= y_0 + 0.1x_0\sqrt{y_0} = 4 + 0.1 \cdot 1\sqrt{4} = 4.2, \\ y_2 &= y_1 + 0.1x_1\sqrt{y_1} = 4.2 + 0.1 \cdot 1.1\sqrt{4.2} \approx 4.42543, \\ y_3 &= y_2 + 0.1x_2\sqrt{y_2} \approx 4.67787, \\ y_4 &\approx 4.95904, \end{aligned}$$

$$y_5 \approx 5.27081.$$

◇

One might wonder how accurate the method is: how good is the approximation? In general, the approximation will improve as the step size decreases, and will decrease as the length of the interval $[x_0, x_n]$ increases. In the following table, we give the Euler approximation, the true value, and the error of the approximation for the IVP in Example 1.

n	Approximation	Exact Value	Error
0	4	4	0
1	4.2	4.21276	0.01276
2	4.42543	4.45210	0.02667
3	4.67787	4.71976	0.04189
4	4.95904	5.01760	0.05856
5	5.27081	5.34766	0.07685

Example 2. Use Euler's method to find approximations to the solution of the IVP

$$y' = y, \quad y(0) = 1$$

at $x = 1$ by taking 1, 2, and 4 steps.

Solution. The formula for Euler's method in this case is $y_{n+1} = y_n + hy_n$, and the step size for N steps is $h = 1/N$. For 1 step, $h = 1$ and $\phi(1) \approx y_1 = 1 + 1 \cdot 1 = 2$. For 2 steps, $h = 1/2$ and we get

$$\begin{aligned} y_1 &= 1 + 0.5(1) = 1.5, \\ \phi(1) \approx y_2 &= 1.5 + 0.5(1.5) = 2.25. \end{aligned}$$

For 4 steps, $h = 1/4$ and we calculate

$$\begin{aligned} y_1 &= 1 + 0.25(1) = 1.25, \\ y_2 &= 1.25 + 0.25(1.25) = 1.5625, \\ y_3 &= 1.5625 + 0.25(1.5625) = 1.953125, \\ \phi(1) \approx y_4 &= 1.953125 + 0.25(1.953125) \approx 2.44141. \end{aligned}$$

◇

Since the solution to the IVP in Example 2 is $\phi(x) = e^x$, we have calculated approximations for $e \approx 2.718281828\dots$. These approximations continue to improve as we increase the number of steps (thereby decreasing step size), as demonstrated in the table below.

N	Approximation of e
1	2
2	2.25
4	2.44141
8	2.56578
16	2.63793

Homework: p. 28 #1, 3, 5-8