Section 2.2: Separable Equations

In Chapter 2 we will focus on developing techniques for solving first-order DE's; subsequent chapters will deal with higher-order DE's. The simplest class of first-order DE's are the separable equations. They are relatively easy to deal with because solving them only requires integration.

Definition. A first-order DE is called separable if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y).$$

Essentially, this means that in the function f(x, y) that represents the derivative, we can "separate" the variables x and y (hence the terminology), writing the multivariable function as a product of two functions of a single variable.

Example 1. The equation $\frac{dy}{dx} = \frac{2x + xy}{y^2 + 1}$ is separable since the right-hand side can be factored as $x\frac{2+y}{y^2+1}$. However, the equation $\frac{dy}{dx} = 1 + xy$ is not separable since there is no factorization of the RHS that will separate the variables.

Now that we know how to recognize a separable DE, how does the method work? We present first the informal heuristic that is actually used in computations. From the form

$$\frac{dy}{dx} = g(x)p(y),$$

multiply both sides by dx and divide by p(y) to obtain

$$h(y)dy = g(x)dx,$$

where we define h(y) = 1/p(y). Then just integrate both sides:

$$\int h(y)dy = \int g(x)dx \Rightarrow H(y) = G(x) + C,$$

where both constants of integration are merged into one. This produces an implicit solution to the DE.

There are two important observations to make. First, there could be additional solutions to the DE that we lose when dividing by p(y); namely, if c is a value such that p(c) = 0, then $y \equiv c$ is also a solution to the DE. Sometimes these solutions can be recovered by allowing the constant of integration C to take the value 0, but other times they do not appear at all in the general solution. This is not necessarily a problem unless one is interested in finding every possible solution to the DE.

A more pressing concern is the phrase "multiply both sides by dx". Remember that the differential $\frac{dy}{dx}$ is equivalent to writing y', and is just a symbol for the derivative, and is NOT an actual fraction. Rigorously speaking, trying to multiply by dx is akin to trying to divide sin x by sin - it makes absolutely no sense. Why then, have I just told you to do it? Because it is an easier way of getting the same end result. Let us show how we would solve a separable DE without resorting to such absurdities as multiplying by dx.

We start with $\frac{dy}{dx} = g(x)p(y) = \frac{g(x)}{h(y)}$ and still multiply both sides by h(y) (this does make sense) to get

$$h(y)\frac{dy}{dx} = g(x).$$

Now define H(y) and G(x) to be the antiderivatives of h and g; that is, H'(y) = h(y), G'(x) = g(x). Substituting this into the previous equation gives

$$H'(y)\frac{dy}{dx} = G'(x).$$

Now observe that the LHS is the result of applying the chain rule to the composite function H(y(x)): $\frac{d}{dx}H(y(x)) = H'(y(x))y'(x) = H'(y(x))\frac{dy}{dx}$. Therefore, integrating both sides of the previous equation yields H(y(x)) = G(x) + C, which is the same thing we got previously since in that case also H'(y) = h(y), G'(x) = g(x).

Doing it the first way avoids the need to reference the composite function H(y(x)), so even though we are abusing notation, we proceed with it, understanding that we are abusing it, simply to make our lives easier. Let us now familiarize ourselves with this particular abuse of notation by doing some examples.

Example 2. Solve the nonlinear equation $\frac{dy}{dx} = \frac{x-5}{y^2}$.

Solution. Separating the variables, we have $y^2 dy = (x-5)dx$. Integrating, we get

$$\int y^2 dy = \int (x-5)dx \Rightarrow \frac{y^3}{3} = \frac{x^2}{2} - 5x + C.$$

For this DE, we can actually solve explicitly for y:

$$y = \left(\frac{3}{2}x^2 - 15x + C_1\right)^{1/3},$$

where $C_1 = 3C$.

Example 3. Solve the initial value problem $\frac{dy}{dx} = \frac{y-1}{x+3}, y(-1) = 0.$

Solution. Again by separating variables we have

$$\frac{dy}{y-1} = \frac{dx}{x+3} \Rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x+3} \Rightarrow \ln|y-1| = \ln|x+3| + C.$$

Now we solve for y explicitly by exponentiating the equation:

$$e^{\ln|y-1|} = e^C e^{\ln|x+3|} \Rightarrow |y-1| = e^C |x+3| = C_1 |x+3|,$$

where $C_1 = e^C$. We can remove the absolute values by introducing a plus or minus: $y - 1 = \pm C_1(x + 3) \Leftrightarrow y = 1 \pm C_1(x + 3)$, where the choice of sign will depend on the values of x and y. We can even do away with the plus or minus by replacing the positive

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constant C_1 with an arbitrary (nonzero) constant K, so that y = 1 + K(x+3). Now we apply the initial condition to obtain a particular solution:

$$0 = 1 + K(-1+3) = 2K + 1 \Rightarrow K = -1/2,$$

so the solution to the IVP is $y = 1 - \frac{1}{2}(x+3) = -\frac{1}{2}(x+1)$.

Remark. After reaching the equation $\ln |y-1| = \ln |x+3| + C$, one could instead immediately substitute the initial condition to solve for C, and then solve for an explicit function of y. Also note that since the value 1 makes p(y) = y - 1 equal to zero, another solution is given by $y \equiv 1$, but this can be recovered by allowing K = 0 in the general solution y = 1 + K(x+3).

Example 4. Solve the nonlinear equation $\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos y + e^y}$.

Solution. Again by separation of variables, we obtain

$$(\cos y + e^y)dy = (6x^5 - 2x + 1)dx \Rightarrow \int (\cos y + e^y)dy = \int (6x^5 - 2x + 1)dx$$
$$\Rightarrow \sin y + e^y = x^6 - x^2 + x + C.$$

This is an implicit solution, but it is impossible to isolate y and get an explicit solution, so this is the best we can do. \diamond

Homework: pp. 43-45 #1-25 odd, 33, 34, 35, 37.

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