Section 2.3: Linear Equations

Introduction: Easy Special Cases

In this section we introduce a technique for solving first-order DE's which are linear. Recall that such equations have the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

The distinguishing characteristic of such equations is that $y, \frac{dy}{dx}$ are only raised to the first power, and their coefficient functions depend only on x, not on y.

There are two special situations in which the solution to a linear DE is very easy. The first occurs when $a_0(x) = 0$; that is, when the DE has the form

$$a_1(x)\frac{dy}{dx} = b(x),$$

or equivalently,

$$\frac{dy}{dx} = \frac{b(x)}{a_1(x)}.$$

Here we obtain the solution simply by integrating: $y(x) = \int \frac{b(x)}{a_1(x)} dx + C$. However, in general a given linear DE cannot be reduced to such a simple form.

The second situation occurs when $a_0(x) = a'_1(x)$. In this case, we can use the product rule to simplify the left-hand side:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \Rightarrow a_1(x)y' + a_1'(x)y = \frac{d}{dx}[a_1(x)y] = b(x).$$

Again, we can solve by just integrating:

$$a_1(x)y = \int b(x)dx + C \Rightarrow y(x) = \frac{1}{a_1(x)} \left[\int b(x)dx + C \right].$$

At first glance, it would seem that this scenario is equally unlikely: most often the coefficient of y will not be the derivative of the coefficient of $\frac{dy}{dx}$. But it turns out that we can convert a general linear DE to one having this form if we multiply the equation by a function $\mu(x)$ called the **integrating factor**. We now proceed to construct this integrating factor.

Derivation of Integrating Factor

Given a linear DE $a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$, write it in standard form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$. Our goal is to find a function $\mu(x)$ so that after multiplying through by this function, this left-hand side follows the product rule. That is, we want

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}[\mu(x)y] = \mu(x)\frac{dy}{dx} + \mu'(x)y.$$

Matching coefficients on both sides of this equation tells us that $\mu(x)$ should satisfy the DE $\mu'(x) = \frac{d\mu}{dx} = \mu(x)P(x)$. Fortunately, this DE is separable (since currently this is the only type of DE we know how to solve), and we have

$$\frac{d\mu}{\mu} = P(x)dx \Rightarrow \ln \mu = \int P(x)dx \Rightarrow \mu(x) = e^{\int P(x)dx}$$

Therefore, the initial DE reduces to $\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$, so the solution is

$$y(x) = \frac{1}{\mu(x)} \left[\int \mu(x)Q(x)dx + C \right].$$

This equation gives the general solution for a linear first-order DE, and since it involves a single constant C, it gives a one-parameter family of solutions.

Let us now practice using the integrating factor to solve linear DE's.

Applying the Integrating Factor to Solve Linear DE's

Example 1. Find the general solution to $\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x, \quad x > 0.$

Solution. First convert to standard form by multiplying by *x*:

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x$$

Next, calculate the integrating factor:

$$\mu(x) = e^{\int P(x)dx} = e^{\int -(2/x)dx} = e^{-2\ln x} = x^{-2}.$$

Multiplying through by the IF gives

$$\frac{d}{dx}(x^{-2}y) = \cos x \Rightarrow x^{-2}y = \sin x + C \Rightarrow y = x^{2}\sin x + Cx^{2}.$$

Example 2. Find the general solution to $\frac{dy}{dx} + 4y = x^2 e^{-4x}$.

Solution. This DE is already in standard form, so we find the integrating factor:

$$u(x) = e^{\int 4dx} = e^{4x}$$

Multiplying through the original equation yields

$$\frac{d}{dx}(e^{4x}y) = x^2,$$

so integrating and dividing gives us

$$e^{4x}y = \frac{1}{3}x^3 + C \Rightarrow y = \frac{1}{3}x^3e^{-4x} + Ce^{-4x}.$$

Example 3. A rock contains two radioactive isotopes: RA_1 which decays into RA_2 , which then decays into stable atoms. The rate at which RA_1 decays into RA_2 is $50e^{-10t}$ kg/sec. Since the rate of decay of RA_2 is proportional to the mass y(t) of RA_2 present, the rate of change in RA_2 is

$$\frac{dy}{dt} = 50e^{-10t} - ky,$$

where k = 2/sec is the decay constant. If the initial mass y(0) = 40 kg, find the mass y(t) for $t \ge 0$.

Solution. Again, we write in standard form: $\frac{dy}{dt} + 2y = 50e^{-10t}$. The integrating factor is $\mu(t) = e^{\int 2dt} = e^{2t}$, so multiplying through by $\mu(t)$ gives

$$\frac{d}{dt}(e^{2t}y) = 50e^{-8t} \Rightarrow e^{2t}y = -\frac{25}{4}e^{-8t} + C \Rightarrow y = -\frac{25}{4}e^{-10t} + Ce^{-2t}.$$

To get the particular solution, we input the initial condition:

$$40 = -\frac{25}{4} + C \Rightarrow C = 40 + \frac{25}{4} = \frac{185}{4}$$

Therefore, the mass of RA_2 present at time t is given by

$$y(t) = \frac{185}{4}e^{-2t} - \frac{25}{4}e^{-10t}.$$

Remark. In real-world applications, sometimes the integral involved in calculating the integrating factor, or the integral of the expression $\mu(x)Q(x)$, cannot be expressed with elementary functions (polynomials, trig functions, exponentials, etc.). In these cases, one must resort to a numerical method such as Euler's method or Simpson's rule in order to approximate values of a solution curve.

Homework: pp. 51-52, #1-21 odd.