Section 2.4: Exact Equations

Introduction: Idea Based on Level Curves

We have seen how to solve first-order DE’s which are either separable or linear. We would like to find other differential forms which are relatively easy to solve. The next type of equation comes from examining level curves.

Recall from multivariable calculus that a level curve of a function of two variables has the form \( F(x, y) = C \) for some value \( C \) in the range of \( F \). To find the slope of the tangent line to a level curve, we apply the multi-dimensional chain rule and solve for \( dy/dx \):

\[
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}.
\]

Formally multiplying the left-hand side of the first equation above by \( dx \) gives an expression we define to be the total differential of \( F \):

\[
dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.
\]

(We say “formally” because, as we stressed earlier, multiplying by the notation \( dx \) is nonsensical.) In this notation, solving \( dF = 0 \) for \( dy/dx \) produces the same slope we just calculated.

The key observation is that \( \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = f(x, y) \) is a DE, and its solution will be exactly the level curve \( F(x, y) = C \). Now every first-order DE \( \frac{dy}{dx} = f(x, y) \) can be rewritten (non-uniquely) in the form

\[
M(x, y)dx + N(x, y)dy = 0;
\]

our work above says that if \( M(x, y) = \frac{\partial F}{\partial x} \) and \( N(x, y) = \frac{\partial F}{\partial y} \) for some function \( F(x, y) \), then the solution to the DE is \( F(x, y) = C \). With this motivation, we present the corresponding definitions.

Definition. The differential form \( M(x, y)dx + N(x, y)dy \) is said to be exact in a rectangle \( R \) if there is a function \( F(x, y) \) such that

\[
\frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y)
\]

for all \( (x, y) \in R \). If \( M(x, y)dx + N(x, y)dy \) is an exact differential form, then

\[
M(x, y)dx + N(x, y)dy = 0
\]

is called an exact equation.
Criterion and Solution to Exact Differential Equations

Now that we have defined exact equations, we need first a test to determine whether a given DE is exact, and secondly a procedure for finding the function $F(x, y)$ mentioned in the definition. Let us focus first on the test for exactness. If we have an exact differential form, then

$$M(x, y) = \frac{\partial F}{\partial x} = F_x \Rightarrow \frac{\partial}{\partial y} M(x, y) = F_{xy}$$

and similarly

$$N(x, y) = \frac{\partial F}{\partial y} = F_y \Rightarrow \frac{\partial}{\partial x} N(x, y) = F_{yx}.$$ 

Recall again from multivariable calculus that by Clairaut’s Theorem, $F_{xy} = F_{yx}$ as long as they are continuous. This implies that $\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$. It turns out that this equality is not only a necessary condition, but also a sufficient condition for exactness (this is proved in the text), so we have the following.

**Theorem 1.** Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle $R$. Then $M(x, y)dx + N(x, y)dy = 0$ is an exact equation in $R$ if and only if

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

holds for all $(x, y) \in R$.

Therefore, we can now recognize whether a DE is exact or not by rewriting it in the differential form $M(x, y)dx + N(x, y)dy = 0$ and checking the “cross-partial” condition in the theorem (recall checking whether a vector field is conservative in multivariable calculus). Speaking of conservative vector fields, the answer to our second question, how to find the function $F(x, y)$, is essentially finding the potential function of a conservative vector field. Let us summarize this procedure in the new context.

Since we know at this point that the DE is exact, we have $\frac{\partial F}{\partial x} = M$. Integrating with respect to $x$ yields

$$F(x, y) = \int M(x, y)dx + g(y),$$

where we write $g(y)$ instead of $C$ because any function involving only $y$ has partial derivative 0 with respect to $x$. To find this function $g$, take the partial derivative with respect to $y$ of the above equation, and use exactness to get

$$N(x, y) = \frac{\partial}{\partial y} \left[ \int M(x, y)dx \right] + g'(y),$$

which we can solve for $g'(y)$. Integrating with respect to $y$ will give an expression for $g(y)$ which we can substitute back into the first equation. Then the implicit solution to the DE is $F(x, y) = C$. 

There is nothing special about starting with \( M \) however; we could just as easily start with \( \partial F/\partial y = \nabla \) and switch the roles of \( x \) and \( y \) in the above procedure. In fact, if both of these integrals are easy to compute, one can bypass finding the “constants of integration” (like \( g(y) \)) by combining the results of both integrals. If one of the integrals is difficult to compute, however, it is easier to follow the method outlined above. We show examples of both ways in the examples that follow.

**Examples**

**Example 1.** Solve the DE \( \frac{dy}{dx} = -\frac{2xy^2 + 1}{2x^2y} \).

**Solution.** First convert to total differential form: \((2xy^2 + 1)dx + (2x^2y)dy = 0\). We apply the test for exactness with \( M(x, y) = 2xy^2 + 1 \), \( N(x, y) = 2x^2y \): \( M_y = 4xy \) and \( N_x = 4xy \), so the DE is exact. Both \( M \) and \( N \) appear easy to integrate, so we find the “potential function” as follows:

\[
\int M(x, y)dx = \int (2xy^2 + 1)dx = x^2y^2 + x + g(y),
\]

\[
\int N(x, y)dy = \int (2x^2y)dy = x^2y^2 + h(x).
\]

If we write down every term that appears in the integrals (without counting repetitions), we get \( F(x, y) = x^2y^2 + x \), so an implicit solution to the DE is \( x^2y^2 + x = C \).

**Example 2.** Solve the DE \((2xy - \sec^2x)dx + (x^2 + 2y)dy = 0\).

**Solution.** This DE is already in total differential form with \( M(x, y) = 2xy - \sec^2x \), \( N(x, y) = x^2 + 2y \). Testing for exactness, we have \( M_y = 2x \), \( N_x = 2x \), so the DE is indeed exact. Again, both functions are easily integrable:

\[
\int M(x, y)dx = \int (2xy - \sec^2x)dx = x^2y - \tan x + g(y),
\]

\[
\int N(x, y)dy = \int (x^2 + 2y)dy = x^2y + y^2 + h(x).
\]

Therefore, we have \( F(x, y) = x^2y - \tan x + y^2 \), and the implicit solution is \( x^2y - \tan x + y^2 = C \).

**Example 3.** Solve the DE \((1 + e^x y + xe^x)dx + (xe^x + 2)dy = 0\).

**Solution.** Again, the DE is in total differential form with \( M(x, y) = 1 + e^x y + xe^x \), \( N(x, y) = xe^x + 2 \). Testing for exactness, we have \( M_y = e^x + xe^x \) and \( N_x = e^x + xe^x \), so the DE is exact. This time, however, the function \( M(x, y) \) looks difficult to integrate with respect to \( x \) because of the \( xe^x \) term. So we begin with

\[
F(x, y) = \int N(x, y)dy = \int (xe^x + 2)dy = xe^xy + 2y + h(x).
\]
Now we take the partial with respect to $x$ of both sides:

$$\frac{\partial F}{\partial x} = M(x, y) = 1 + e^x y + x e^x y = e^x y + x e^x y + h'(x),$$

which implies $h'(x) = 1$ and thus $h(x) = x$. So we have $F(x, y) = x e^x y + 2y + x$ and an implicit solution is $x e^x y + 2y + x = C$, which actually can be solved explicitly:

$$y = (C - x)/(xe^x + 2).$$

Of course, the vast majority of DE’s will not be exact. However, through clever multiplication (such as that introduced in Section 2.3 for linear equations), we can manipulate nonexact DE’s into exact DE’s. The last example provides an appetizer for techniques to be explored in the next section.

**Example 4.** Show that $(x + 3x^3 \sin y)dx + (x^4 \cos y)dy = 0$ is not exact, but that multiplying this equation by the factor $x^{-1}$ yields an exact equation. Use this fact to solve the original DE.

**Solution.** We test for exactness with $M(x, y) = x + 3x^3 \sin y, N(x, y) = x^4 \cos y$: $M_y = 3x^3 \cos y$, but $N_x = 4x^3 \cos y$, so the given DE is not exact. Once we multiply the equation, we get

$$(1 + 3x^2 \sin y)dx + (x^3 \cos y)dy = 0,$$

and here $M(x, y) = 1 + 3x^2 \sin y, N(x, y) = x^3 \cos y$. Re-testing for exactness, we have $M_y = 3x^2 \cos y, N_x = 3x^2 \cos y$, so the new DE is exact. To find the solution, integrate:

$$\int M(x, y)dx = \int (1 + 3x^2 \sin y)dx = x + x^3 \sin y + g(y),$$

$$\int N(x, y)dy = \int (x^3 \cos y)dy = x^3 \sin y + h(x),$$

so we get $F(x, y) = x + x^3 \sin y$ and an implicit solution is $x + x^3 \sin y = C$. Since the two DE’s differ only by a factor of $x$, a solution to one will also be a solution to the other whenever $x \neq 0$. Therefore, the solution to the original DE is also $x + x^3 \sin y = C$. ♦

Homework: pp. 61-62, #1-29 odd.