

## Section 2.5: Special Integrating Factors

### Generalizing Integrating Factors

We resume our discussion from the previous section about how to manipulate DE's which are not exact into exact equations. To begin, consider the standard form of a linear first-order DE (discussed in Section 2.3)

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Rewriting in differential form through formal multiplication by  $dx$ , we get

$$[P(x)y - Q(x)]dx + dy = 0.$$

If this equation were exact, we would have  $P(x) = 0$ , which is not generally true. However, let us multiply the equation by the integrating factor  $\mu(x) = e^{\int P(x)dx}$ :

$$[\mu(x)P(x)y - \mu(x)Q(x)]dx + \mu(x)dy = 0.$$

Now the exactness condition is  $\mu(x)P(x) = \mu'(x)$ , which you may recall is exactly how we derived the formula for the integrating factor, which means that the new equation is exact! We can now revise our earlier concept of this object.

**Definition.** If the equation

$$(1) \quad M(x, y)dx + N(x, y)dy = 0$$

is not exact, but the equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact, then  $\mu(x, y)$  is an integrating factor of equation (1).

**Example 1.** Show that  $\mu(x, y) = xy^2$  is an integrating factor for

$$(2y - 6x)dx + (3x - 4x^2y^{-1})dy = 0,$$

and use this integrating factor to solve the DE.

**Solution.** We find  $M_y = 2, N_x = 3 - 8xy^{-1}$ , so the original equation is not exact. Multiplying through by  $\mu(x, y)$  gives

$$(2xy^3 - 6x^2y^2)dx + (3x^2y^2 - 4x^3y)dy = 0.$$

Re-checking the exactness condition, we have  $M_y = 6xy^2 - 12x^2y, N_x = 6xy^2 - 12x^2y$ , so the new equation is exact, verifying that  $\mu(x, y)$  is an integrating factor for the original equation. We now solve the DE:

$$\int M(x, y)dx = \int (2xy^3 - 6x^2y^2)dx = x^2y^3 - 2x^3y^2 + g(y),$$

$$\int N(x, y)dy = \int (3x^2y^2 - 4x^3y)dy = x^2y^3 - 2x^3y^2 + h(x).$$

Therefore, an implicit solution is  $x^2y^3 - 2x^3y^2 = C$ .

Note that it is possible to lose or gain solutions when multiplying by an integrating factor. In this case,  $y \equiv 0$  is a solution of the revised equation but *not* of the original equation. This extraneous solution appears because when multiplying by  $\mu = xy^2$ , we are actually multiplying by zero if  $y \equiv 0$ . In general, you should always check whether any solutions to  $\mu(x, y) = 0$  are also solutions to the original DE.  $\diamond$

### Finding Integrating Factors

So if we can determine an integrating factor for a nonexact DE, we can convert it to an exact DE and easily solve it. The question which remains is: how can we find an integrating factor? Since we know that the equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact, the condition for exactness must be satisfied; that is,

$$\frac{\partial}{\partial y}[\mu(x, y)M(x, y)] = \frac{\partial}{\partial x}[\mu(x, y)N(x, y)].$$

Applying the product rule and some algebra gives the partial differential equation

$$(2) \quad M \frac{\partial \mu}{\partial y} - N \frac{\partial \mu}{\partial x} = \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mu,$$

which usually is more difficult to solve than the equation we first started with. We say “usually” because there are two exceptions which will help us out.

The exceptions are when the integrating factor is a function of just one variable, not two. Suppose first that  $\mu(x)$  depends only on  $x$ . Then the equation (2) reduces to

$$(3) \quad \frac{\partial \mu}{\partial x} = \left( \frac{\partial M/\partial y - \partial N/\partial x}{N} \right) \mu,$$

which is separable if  $(\partial M/\partial y - \partial N/\partial x)/N$  is a function of  $x$  alone. Similarly, if  $\mu(y)$  depends only on  $y$ , then (2) reduces to

$$\frac{\partial \mu}{\partial y} = \left( \frac{\partial N/\partial x - \partial M/\partial y}{M} \right) \mu,$$

which again is separable if  $(\partial N/\partial x - \partial M/\partial y)/M$  is a function of  $y$  alone.

In these cases, we can solve these separable equations for  $\mu$ ; if we solve equation (3) we obtain

$$\mu(x) = \exp \left[ \int \left( \frac{\partial M/\partial y - \partial N/\partial x}{N} \right) dx \right].$$

The following theorem summarizes our derivations.

**Theorem 1.** *If  $(\partial M/\partial y - \partial N/\partial x)/N$  is continuous and depends only on  $x$ , then*

$$(4) \quad \mu(x) = \exp \left[ \int \left( \frac{\partial M/\partial y - \partial N/\partial x}{N} \right) dx \right]$$

*is an integrating factor for the DE (1).*

If  $(\partial N/\partial x - \partial M/\partial y)/M$  is continuous and depends only on  $y$ , then

$$(5) \quad \mu(y) = \exp \left[ \int \left( \frac{\partial N/\partial x - \partial M/\partial y}{M} \right) dy \right]$$

is an integrating factor for the DE (1).

Of course, several DE's which do not satisfy the hypotheses of this Theorem will also have integrating factors, but explicit formulas for them are difficult to obtain since they will generally depend on both  $x$  and  $y$ . However, the theorem does suggest a method for solving the DE  $Mdx + Ndy = 0$  which summarizes Chapter 2 so far.

- Check whether the DE is separable or linear.
- If not, check whether  $M_y = N_x$  and the DE is exact.
- If not, consider  $\frac{\partial M/\partial y - \partial N/\partial x}{N}$ ; if this is a function only of  $x$ , then an integrating factor is given by formula (4).
- If not, consider  $\frac{\partial N/\partial x - \partial M/\partial y}{M}$ ; if this is a function only of  $y$ , then an integrating factor is given by formula (5).

**Example 2.** Solve the DE  $(2x^2 + y)dx + (x^2y - x)dy = 0$ .

**Solution.** It can be shown that the DE is neither separable nor linear. Checking for exactness,  $M_y = 1$ ,  $N_x = 2xy - 1$ , so it is not exact either. We find

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{1 - (2xy - 1)}{x^2y - x} = \frac{2(1 - xy)}{-x(1 - xy)} = \frac{-2}{x}.$$

Since this is a function of  $x$  alone, we can find an integrating factor using formula (4):

$$\mu(x) = \exp \left( \int \frac{-2}{x} dx \right) = e^{-2 \ln x} = x^{-2}.$$

Multiplying the original DE through by the IF yields

$$(2 + yx^{-2}) + (y - x^{-1})dy = 0,$$

and re-checking exactness, we have  $M_y = x^{-2}$ ,  $N_x = x^{-2}$ , so the new equation is exact, as desired. We integrate:

$$\begin{aligned} \int M(x, y)dx &= \int (2 + yx^{-2})dx = 2x - yx^{-1} + g(y), \\ \int N(x, y)dy &= \int (y - x^{-1})dy = \frac{1}{2}y^2 - yx^{-1} + h(x). \end{aligned}$$

So an implicit solution is  $2x - yx^{-1} + \frac{1}{2}y^2 = C$ . Since the solution  $x \equiv 0$  makes the IF undefined, that does not appear in the implicit solution, but it does solve the initial DE, so we add it to the set of solution curves given by the implicit equation.  $\diamond$

**Example 3.** Solve the DE  $(2y^2 + 2y + 4x^2)dx + (2xy + x)dy = 0$ .

**Solution.** Checking for exactness, we have  $M_y = 4y + 2, N_x = 2y + 1$ ; therefore, the equation is not exact. However,

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{(4y + 2) - (2y + 1)}{2xy + x} = \frac{2y + 1}{x(2y + 1)} = \frac{1}{x}$$

is a function of  $x$  alone. So we have the integrating factor  $\mu(x) = e^{\int(1/x)dx} = e^{\ln x} = x$ . After multiplying the original DE by the IF, we have

$$(2xy^2 + 2xy + 4x^3)dx + (2x^2y + x^2)dy = 0.$$

The new equation is exact since  $M_y = N_x = 4xy + 2x$ . Now integrate  $M$  and  $N$ :

$$\int M(x, y)dx = \int (2xy^2 + 2xy + 4x^3)dx = x^2y^2 + x^2y + x^4 + g(y),$$

$$\int N(x, y)dy = \int (2x^2y + x^2)dy = x^2y^2 + x^2y + h(x).$$

Therefore, an implicit solution is  $x^2y^2 + x^2y + x^4 = C$ . ◇

Homework: p. 67, #1-13 odd.