Section 2.6: Substitutions and Transformations

Homogeneous Equations

In the previous section, we examined how to convert an equation that we had no method of solving into an exact equation, which we can solve. There are several other types of equations that can be transformed into separable or linear equations through a suitable substitution.

Definition. If the right-hand side of the DE

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

can be expressed as a function solely of the ratio y/x, then we say the equation is homogeneous.

To solve a homogeneous equation, we make the natural substitution $v = \dfrac{y}{x}$. This implies that $y = xv$ and thus $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Therefore, the equation (1) can be written as

$$v + x\frac{dv}{dx} = G(v),$$

which is separable.

Example 1. Solve $(xy + y^2 + x^2)dx - x^2dy = 0$.

Solution. It can be checked that the equation is not separable, linear, or exact. Rewrite the DE as

$$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1.$$

Since the RHS is a function of y/x, the DE is homogeneous. Therefore, let $v = y/x$ to get

$$v + x\frac{dv}{dx} = v^2 + v + 1.$$

Separating variables and integrating yields

$$\int \frac{dv}{v^2 + 1} = \int \frac{dx}{x} \Rightarrow \arctan v = \ln|x| + C,$$

and therefore $v = \tan(\ln|x| + C)$. Going back to y gives $y = x\tan(\ln|x| + C)$ as an explicit solution; an additional solution is $x \equiv 0$. ◊

Example 2. Solve $(x^2 + y^2)dx + 2xydy = 0$.

Solution. We rewrite the DE as $\dfrac{dy}{dx} = -\dfrac{x^2 + y^2}{2xy} = -\dfrac{1}{2}\left(\dfrac{x}{y} + \dfrac{y}{x}\right)$. Since the RHS is a function of y/x, the DE is homogeneous, so let $v = y/x$. Then

$$v + x\frac{dv}{dx} = -\frac{1}{2}(v^{-1} + v) \Rightarrow x\frac{dv}{dx} = -\frac{1}{2}v^{-1} - \frac{3}{2}v = \frac{-1 - 3v^2}{2v}.$$

Separating variables and integrating gives

$$\int \frac{-2v dv}{1+3v^2} = \int \frac{dx}{x}$$

$$-\frac{1}{3}\ln(1+3v^2) = \ln|x| + C \Rightarrow -\frac{1}{3}\ln\left(1+\frac{3y^2}{x^2}\right) = \ln|x| + C,$$

in addition to the solution $x \equiv 0$. $\Diamond$

Bernoulli Equations

Definition. A first-order equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \tag{2}$$

where $P(x), Q(x)$ are continuous on an interval (a,b) and $n \in \mathbb{R}$, is called a Bernoulli equation.

If $n = 0$ or 1, then a Bernoulli equation is also linear, and can be solved by using an integrating factor as in Section 2.3. For other values of n, we make the substitution $v = y^{1-n}$ which transforms the Bernoulli equation into a linear equation. Indeed, dividing equation (2) by y^n gives

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x).$$

Since $v = y^{1-n}$ implies $\dfrac{dv}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$, we have

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x),$$

which is linear in v.

Example 3. Solve $\dfrac{dy}{dx} - 5y = -\dfrac{5}{2}xy^3$.

Solution. This is a Bernoulli equation with $P(x) = -5, Q(x) = -\dfrac{5}{2}x$, and $n = 3$. First divide by y^3 to get

$$y^{-3}\frac{dy}{dx} - 5y^{-2} = -\frac{5}{2}x.$$

Next substitute $v = y^{1-3} = y^{-2} \Rightarrow \dfrac{dv}{dx} = -2y^{-3}\dfrac{dy}{dx}$ to obtain

$$-\frac{1}{2}\frac{dv}{dx} - 5v = -\frac{5}{2}x \Rightarrow \frac{dv}{dx} + 10v = 5x.$$

Now we find the integrating factor $\mu(x) = e^{\int 10dx} = e^{10x}$, and multiplying on both sides gives

$$\frac{d}{dx}[e^{10x}v] = 5xe^{10x}.$$

Integrating, we have

$$e^{10x}v = \frac{1}{2}xe^{10x} - \frac{1}{20}e^{10x} + C \Rightarrow v = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}.$$

Going back to the original variable y gives us

$$y^{-2} = \frac{x}{2} - \frac{1}{20} + Ce^{-10x}.$$

This equation does not include the additional solution $y \equiv 0$, which we lost when dividing by y^3 at the very beginning. $\Diamond$

Example 4. Solve $\dfrac{dy}{dx} - y = e^{2x}y^3$.

Solution. This is a Bernoulli equation with $n = 3$, so divide the equation by y^{-3}:

$$y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}.$$

Next, let $v = y^{-2}$, $\dfrac{dv}{dx} = -2y^{-3}\dfrac{dy}{dx}$; substituting these gives

$$-\frac{1}{2}\frac{dv}{dx} - v = e^{2x} \Rightarrow \frac{dv}{dx} + 2v = -2e^{2x}.$$

Now the integrating factor is $\mu(x) = e^{\int 2dx} = e^{2x}$, so multiplying through we get

$$\frac{d}{dx}[e^{2x}v] = -2e^{4x}.$$

Now integrate and solve:

$$e^{2x}v = -\frac{1}{2}e^{4x} + C \Rightarrow v = -\frac{1}{2}e^{2x} + Ce^{-2x}.$$

Finally, replace v with y^{-2}:

$$y^{-2} = -\frac{1}{2}e^{2x} + Ce^{-2x} \Rightarrow y^2 = \frac{-2}{e^{2x} + Ce^{-2x}}.$$

This equation does not include the additional solution $y \equiv 0$, which we lost when dividing by y^3 at the very beginning. $\Diamond$

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