Section 4.1: The Mass-Spring Oscillator

Developing the Differential Equation

We have explored various techniques for solving first-order DE's in Chapter 2; now we move to consider DE's of higher order in Chapters 4 and 6. Chapter 4 will focus on linear second-order equations. Indeed, a frequently encountered DE involves applications of Newton's second law F = ma; since acceleration is the second derivative of position with respect to time, this is in fact a second-order DE.

A particular application, which shall serve for us as a motivating example, is the massspring oscillator: a mass m attached to a spring which is fixed at one end. We work to develop a DE which describes the motion of the oscillator and accounts for forces due to the spring elasticity, damping friction, and external forces. When the mass is still and the spring unstretched, the system is at equilibrium, and we measure the coordinate of the mass by its displacement from the equilibrium position.

One force acting on the mass is the spring force: as the spring is stretched and compressed, it exerts a force resisting the direction of displacement. Hooke's law states that this force is directly proportional to the displacement:

$$F_{spring} = -ky,$$

where k > 0 is the stiffness constant and the sign indicates that the force opposes displacement.

A second force in this system is friction; this is typically modeled as being proportional to velocity:

$$F_{friction} = -by',$$

where $b \ge 0$ is called the damping coefficient and the sign again indicates that the force opposes displacement. We will collect all the remaining forces which are external to the system into a single function $F_{ext}(t)$, which is assumed to be known. We can now use Newton's law to write

(1)
$$my'' + by' + ky = F_{ext}(t).$$

Damping

In the remainder of this section, we see how our intuition about this system under various scenarios aligns with the actual solutions. We defer demonstrating how to find such solutions to subsequent sections. The simplest scenario is when we assume there is no damping/friction (i.e. b = 0) and no external force. In this case, we expect the displacement to oscillate perpetually with no drop-off, resembling a sine or cosine graph. The oscillations should speed up with stiffer springs but slow down for heavier masses. The following example confirms this intuition.

Example 1. Verify that if b = 0 and $F_{ext}(t) = 0$, equation (1) has a solution of the form $y(t) = \cos(\omega t)$ and that the angular frequency ω increases with k and decreases with m.

Solution. Equation (1) under these conditions is simplified to

$$my'' + ky = 0.$$

We calculate $y'(t) = -\omega \sin(\omega t), y''(t) = -\omega^2 \cos(\omega t)$, so substituting this gives $-m\omega^2 \cos(\omega t) + k \cos(\omega t) = 0 \Rightarrow \cos(\omega t)(k - m\omega^2) = 0.$

Setting the second factor equal to zero yields $m\omega^2 = k \Rightarrow \omega = \sqrt{k/m}$. Therefore, $y = \cos(\sqrt{k/m}t)$ is a solution, and the angular frequency ω increases as k does, but decreases as m increases.

When damping is present, the oscillations eventually die out, so the solution will take on a different shape. Systems with low damping, such as tuning forks, continue to vibrate for some time, while those with high damping, like car suspension systems, use friction to help suppress or eliminate oscillations. Sometimes this distinction is described as underdamped versus overdamped system.

Example 2. Verify that the damped sinusoid $y(t) = e^{-3t} \cos(4t)$ is a solution to equation (1) if $F_{ext} = 0, m = 1, k = 25$, and b = 6.

Solution. Substituting given values into equation (1) yields the DE

$$y'' + 6y' + 25y = 0$$

We calculate

$$y'(t) = -3e^{-3t}\cos(4t) - 4e^{-3t}\sin(4t)$$

 $y''(t) = 9e^{-3t}\cos(4t) + 12e^{-3t}\sin(4t) + 12e^{-3t}\sin(4t) - 16e^{-3t}\cos(4t) = -7e^{-3t}\cos(4t) + 24e^{-3t}\sin(4t).$ Substituting into the above DE, we get

$$-7e^{-3t}\cos(4t) + 24e^{-3t}\sin(4t) + 6(-3e^{-3t}\cos(4t) - 4e^{-3t}\sin(4t)) + 25(e^{-3t}\cos(4t)) = 0.$$

Example 3. Verify that the exponential function $y(t) = e^{-5t}$ is a solution to equation (1) if $F_{ext} = 0, m = 1, k = 25$, and b = 10.

Solution. The DE to solve is y'' + 10y' + 25y = 0. We find $y'(t) = -5e^{-5t}$, $y''(t) = 25e^{-5t}$. Substituting gives us

$$25e^{-5t} + 10(-5e^{-5t}) + 25e^{-5t} = 0.$$

External Forces

We now make some remarks about systems which are affected by external forces. Typically, such forces are sinusoidal with fixed amplitude, and the resulting solution curves will eventually synchronize with these forces and oscillate at the same frequency, though the initial behavior can be erratic. Examples of such systems are sound system speakers, electronic amplifier circuits, and ocean tides (affected by the moon). It is important to observe that some systems are incredibly sensitive to the resonant frequency: this is the explanation for how perfectly tuned notes can shatter crystal and wind-driven vibrations can destroy bridges. This is a major point to consider when engineers are designing systems, whether you want to avoid such resonant responses or not (as with radio signals). We simply note that an example of such a solution curve is the form $y(t) = A \cos(\Omega t) + B \sin(\Omega t)$, where Ω is the driving frequency.

The majority of this chapter will focus on DE's of the form

$$ay'' + by' + cy = f(t),$$

where a, b, c are constants and f(t) is a given function. These are called linear second-order ODE's with constant coefficients.

Homework: pp. 157-158, #2-5 all.