Section 4.2: Homogeneous Linear Equations: The General Solution

The Auxiliary Equation

Recall that linear second-order DE's with constant coefficients have the form

$$ay'' + by' + cy = f(t);$$

we shall begin by considering the case where $f(t) = 0$ (this corresponds to a mass-spring oscillator vibrating without being influenced by external forces). Our DE now has the form

$$ay'' + by' + cy = 0, \tag{1}$$

which is called *homogeneous*. (Frustratingly, this terminology does not imply any similarity or tie with the homogeneous equations studied in Section 2.6.)

A preliminary observation about this DE is that a solution must have the property that its second derivative can be written as a linear combination of its first derivative and the function itself. Since the exponential function behaves in this way, we find a solution of the form $y = e^{rt}$:

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c) = 0 \Rightarrow$$

$$ar^2 + br + c = 0, \tag{2}$$

since the exponential function is never zero. This calculation shows that $y = e^{rt}$ is a solution to our DE (1) exactly when r is a root of equation (2), which is called the auxiliary equation (or the characteristic equation) associated with the homogeneous equation (1).

Of course, r can be found simply by using the quadratic formula. Recall from precalculus algebra that if the discriminant $b^2 - 4ac$ is positive, the equation (2) has two distinct real roots; if $b^2 - 4ac = 0$, there is a repeated real root; and if $b^2 - 4ac < 0$, the roots are complex conjugates. We focus on the real case in this section, and handle the complex case in Section 4.3.

Example 1. Find a pair of solutions to $y'' + 5y' - 6y = 0$.

Solution. The auxiliary equation for this DE is

$$r^2 + 5r - 6 = 0 \Rightarrow (r - 1)(r + 6) = 0,$$

which has roots $r_1 = 1, r_2 = -6$. Therefore, two solutions are $y_1 = e^t$ and $y_2 = e^{-6t}$. ♢

The General Solution

One very nice property of the homogeneous equation (1) is that if we find two solutions $y_1(t)$ and $y_2(t)$, any linear combination $y(t) = c_1y_1(t) + c_2y_2(t)$ of these will also be a solution to (1). Indeed,

$$\begin{aligned} ay'' + by' + c &= a(c_1y_1'' + c_2y_2'') + b(c_1y_1' + c_2y_2') + c(c_1y_1 + c_2y_2) \\ &= c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) \\ &= 0 + 0 = 0 \end{aligned}$$

Since there are two constants now (because we are solving a *second*-order DE), if we want a particular solution we will need to be given *two* initial conditions, rather than just one.

Example 2. Solve the IVP $y'' + 2y' - y = 0; \quad y(0) = 0, \;\; y'(0) = -1.$

Solution. First we find two solutions as in Example 1. The auxiliary equation is

$$r^2 + 2r - 1 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}.$$

Therefore, solutions of the DE have the form

$$y(t) = c_1e^{(-1+\sqrt{2})t} + c_2e^{(-1-\sqrt{2})t}.$$

To find the particular solution, we take the derivative and obtain a system of equations from the initial conditions:

$$y(0) = c_1 + c_2 = 0, \tag{3}$$

$$y'(0) = (-1+\sqrt{2})c_1 + (-1-\sqrt{2})c_2 = -1. \tag{4}$$

Therefore, $c_2 = -c_1$ from equation (3), and substituting this into equation (4) gives

$$(-1+\sqrt{2})c_1 - (-1-\sqrt{2})c_1 = 2\sqrt{2}c_1 = -1 \Rightarrow c_1 = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4},$$

and thus $c_2 = \frac{\sqrt{2}}{4}$. So the solution to the IVP is

$$y(t) = -\frac{\sqrt{2}}{4}e^{(-1+\sqrt{2})t} + \frac{\sqrt{2}}{4}e^{(-1-\sqrt{2})t}. \qquad \Diamond$$

We have seen that linear combinations of two solutions are also solutions to (1), but are there any solutions that do not come about in this way? It turns out that the answer is no, but before making that statement, we introduce a needed definition.

Definition. A pair of functions $y_1(t)$ and $y_2(t)$ is said to be linearly independent on the interval I if neither of them is a constant multiple of the other on the whole interval I. If one of them *is* a constant multiple of the other on the whole interval I, we say they are linearly dependent on I.

Theorem 1. *If $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the DE (1) on $(-\infty, \infty)$, then any solution of (1) has the form*

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

The proof of this is somewhat technical; interested students can consult the text (pp. 161-162). However, it tells us that once we find two linearly independent solutions, we have really found them all. This is great in the case that the auxiliary equation has two distinct real roots, because $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are linearly independent when $r_1 \neq r_2$ (because they are not constant multiples of each other). However, if the auxiliary equation has a single repeated root, we only get one solution, and to apply Theorem 1 we need to find another linearly independent solution.

It turns out that $y_2(t) = te^{rt}$ is what we want. To see why, we find the derivatives

$$y_2'(t) = e^{rt} + rte^{rt}, y_2''(t) = re^{rt} + re^{rt} + r^2te^{rt} = 2re^{rt} + r^2te^{rt}.$$

Then substituting into the DE (1) gives

$$ay_2'' + by_2' + cy_2 = 2are^{rt} + ar^2te^{rt} + be^{rt} + brte^{rt} + cte^{rt} = [2ar + b]e^{rt} + [ar^2 + br + c]te^{rt}.$$

Since r is a root of the auxiliary equation, the second term is zero. Moreover, since r is a double root, we have $r = \dfrac{-b}{2a} \Rightarrow 2ar + b = 0$, so in this case the first term is also zero. This proves that it is a solution to (1), and it is also linearly independent of e^{rt} since they are not constant multiples.

Example 3. Solve the IVP $y'' + 4y' + 4y = 0; \quad y(0) = 1, \ \ y'(0) = 3.$

Solution. The auxiliary equation is $r^2 + 4r + 4 = 0 \Rightarrow (r + 2)^2 = 0$, so it has the single root $r = -2$. Therefore, a general solution to the DE is given by

$$y(t) = c_1e^{-2t} + c_2te^{-2t}.$$

To find the particular solution, use the initial conditions:

$$y(0) = c_1 = 1, \tag{5}$$

$$y'(0) = -2c_1 + c_2 = 3. \tag{6}$$

Substituting $c_1 = 1$ into equation (6) gives $c_2 = 5$, so the solution to the IVP is $y(t) = e^{-2t} + 5te^{-2t}$. ♢

The method can be extended to higher-order DE's by taking linear combinations of n linearly independent solutions to an order n DE. This will be explored in greater detail in Chapter 6.

Homework: p. 165 #1-21 odd, 27-31 odd.