Section 4.3: Auxiliary Equations with Complex Roots

Euler's Formula

In the previous section, we saw that the homogeneous DE

$$ay'' + by' + cy = 0 \tag{1}$$

with auxiliary equation $ar^2 + br + c = 0$ had the general solution $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where r_1, r_2 are the roots of the auxiliary equation. In this section we seek to understand the situation when these roots are complex. When the discriminant $b^2 - 4ac < 0$, the roots of the auxiliary equation are the complex conjugates $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where $\alpha = -\dfrac{b}{2a}$ and $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$ are real numbers. If we follow the same logic as before, one solution to the DE is $e^{(\alpha + i\beta)t}$, but at present we have no way of understanding what this means. Assuming that the law of exponents applies to complex numbers (it does), we can simplify this expression slightly to $e^{\alpha t} e^{i\beta t}$, but we still need to understand the second factor.

For this purpose, let's recall some well-known Maclaurin series from Calc 2 (they are the same for complex numbers as they are for real numbers):

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \\ \cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ... \\ \sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ... \end{aligned}$$

Now we'll apply the Maclaurin series for $e^{i\theta}$ for $\theta \in \mathbb{R}$:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + ... \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + ... \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ...\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

The equation $e^{i\theta} = \cos\theta + i\sin\theta$ is called **Euler's formula**. Using this formula, we can now write the solution to our DE in terms of familiar real functions as

$$e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)).$$

Applying the same formula to $e^{(\alpha - i\beta)t}$, a general solution to the DE when the auxiliary equation has complex roots $\alpha \pm i\beta$ is

$$y(t) = c_1 e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) + c_2 e^{\alpha t}(\cos(\beta t) - i\sin(\beta t)).$$

Example 1. Solve the IVP $y'' + 2y' + 2y = 0; \quad y(0) = 0,\ y'(0) = 2.$

Solution. The auxiliary equation is $r^2 + 2r + 2 = 0$, which has roots

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

Therefore, we have $\alpha = -1, \beta = 1$, so the general solution is

$$y(t) = c_1 e^{-t}(\cos t + i \sin t) + c_2 e^{-t}(\cos t - i \sin t).$$

Taking the derivative gives

$$y'(t) = -c_1 e^{-t}(\cos t + i \sin t) + c_1 e^{-t}(-\sin t + i \cos t) - c_2 e^{-t}(\cos t - i \sin t) + c_2 e^{-t}(-\sin t - i \cos t),$$

so using the initial conditions generates the system

$$y(0) = c_1 + c_2 = 0, \tag{2}$$

$$y'(0) = (-1+i)c_1 + (-1-i)c_2 = 2. \tag{3}$$

Substituting $c_2 = -c_1$ into equation (3) gives

$$(-1+i)c_1 + (1+i)c_1 = 2 \Rightarrow 2ic_1 = 2 \Rightarrow c_1 = \frac{1}{i} = -i,$$

so $c_2 = i$ and we have the particular solution

$$y(t) = -ie^{-t}(\cos t + i \sin t) + ie^{-t}(\cos t - i \sin t) = 2e^{-t} \sin t. \qquad \Diamond$$

Simplified General Solution

Observe that we were able to simplify the solution in Example 1 to a real-valued function that did not involve i. One might wonder whether this is simply a coincidence, or if this always happens. We can write any complex-valued function as $z(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real-valued functions. The derivatives of this function are $z' = u' + iv', z'' = u'' + iv''$. The following theorem tells us that the complex-valued solution $e^{(\alpha+i\beta)t}$ actually yields two linearly independent real-valued solutions.

Theorem 1. *Let $z(t) = u(t) + iv(t)$ be a solution to the DE (1), where $a, b, c \in \mathbb{R}$. Then the real part $u(t)$ and the imaginary part $v(t)$ are real-valued solutions of (1).*

Proof. Since $z(t)$ is a solution, we have $az'' + bz' + cz = 0$. Using the derivatives calculated above, this implies

$$\begin{aligned} a(u'' + iv'') + b(u' + iv') + c(u + iv) &= 0 \Rightarrow \\ (au'' + bu' + cu) + i(av'' + bv' + cv) &= 0 \Rightarrow \end{aligned}$$

$$au'' + bu' + cu = 0, av'' + bv' + cv = 0,$$

so that u and v are both solutions to (1). $\square$

Applying Theorem 1 to the solution $e^{(\alpha+i\beta)t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t)$, we get the following.

Corollary 2. *If the auxiliary equation to DE (1) has complex conjugate roots $\alpha \pm i\beta$, then two linearly independent solutions are $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$. Hence, a general solution is*

$$y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t).$$

Remark. We got these linearly independent real-valued solutions from just one complex solution $e^{(\alpha+i\beta)t}$. However, using the other complex solution $e^{(\alpha-i\beta)t}$ gives the same two real-valued solutions. As an additional comment, we have assumed several properties of complex-valued functions, such as the law of exponents and derivatives; formal justification for these properties can be found in an introductory course in complex analysis.

Example 2. Find a general solution to $y'' + 2y' + 4y = 0$.

Solution. The auxiliary equation is $r^2 + 2r + 4 = 0$ with roots

$$r = \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm i2\sqrt{3}}{2} = -1 \pm i\sqrt{3},$$

so we have $\alpha = -1, \beta = \sqrt{3}$. Therefore, a general solution is

$$y(t) = c_1 e^{-t}\cos(t\sqrt{3}) + c_2 e^{-t}\sin(t\sqrt{3}). \qquad \Diamond$$

Connection to Mass-Spring Oscillators

Recall that the position $y(t)$ of the mass m in a mass-spring oscillator satisfies the DE

$$my''(t) + by'(t) + ky(t) = 0,$$

where b is the damping coefficient and k is the stiffness of the spring.

Example 3. Find the equation of motion for a spring system when $m = 36$ kg, $b = 12$ kg/sec, $k = 37$ kg/sec^2, $y(0) = 0.7$m, and $y'(0) = 0.1$m/sec. Also find the displacement after 10 seconds.

Solution. We want to solve the IVP $36y'' + 12y' + 37y = 0; y(0) = 0.7, y'(0) = 0.1$. The auxiliary equation is $36r^2 + 12r + 37 = 0$, which has roots

$$r = \frac{-12 \pm \sqrt{144 - 4(36)(37)}}{72} = \frac{-12 \pm 12\sqrt{-36}}{72} = \frac{-12 \pm 72i}{72} = -\frac{1}{6} \pm i.$$

Therefore, we have $\alpha = -\frac{1}{6}, \beta = 1$, so the general solution is

$$y(t) = c_1 e^{-t/6}\cos t + c_2 e^{-t/6}\sin t$$

with derivative

$$y'(t) = -\frac{1}{6}c_1 e^{-t/6}\cos t - c_1 e^{-t/6}\sin t - \frac{1}{6}c_2 e^{-t/6}\sin t + c_2 e^{-t/6}\cos t.$$

The initial conditions generate the system

$$\text{(4)} \qquad y(0) = c_1 = 0.7,$$

$$\text{(5)} \qquad y'(0) = -\frac{1}{6}c_1 + c_2 = 0.1$$

Substituting equation (4) into equation (5) gives $c_2 = 0.1 + 0.7/6 = 1.3/6$. Therefore, the motion of the mass is given by

$$y(t) = 0.7e^{-t/6}\cos t + \frac{1.3}{6}e^{-t/6}\sin t$$

and the displacement after 10 seconds is

$$y(10) = 0.7e^{-5/3}\cos 10 + \frac{1.3}{6}e^{-5/3}\sin 10 \approx -0.1332. \qquad \Diamond$$

Given any second-order DE of the form (1), we can interpret it as a mass-spring system with mass a, damping coefficient b, and spring stiffness c, which makes sense physically if $a > 0$ and $b, c \geq 0$. In this case, we expect damped oscillatory solutions as in Example 3. However, since the angular frequency is given by $\beta = \dfrac{\sqrt{4mk - b^2}}{2m}$, oscillations will not occur if $b \geq \sqrt{4mk}$, making the term under the radical negative. Such systems with large damping coefficients are called *overdamped*. An example is the DE $y'' + 4y' + 4y = 0$ with solution $y(t) = c_1e^{-2t} + c_2te^{-2t}$.

We can also predict solutions using the mass-spring analogy when the coefficients b and c are negative. If the damping coefficient is negative, then the friction force imparts energy to the system rather than draining it, so we expect the oscillations to increase in amplitude rather than decrease. An example of such a situation is the DE $36y'' - 12y' + 37y = 0$ with solution

$$y(t) = c_1e^{t/6}\cos t + c_2e^{t/6}\sin t.$$

If the spring stiffness is negative, this means that as a mass moves away from the equilibrium position, the spring repels the mass farther rather than pulling/pushing it back to equilibrium. Therefore, we expect the solutions to approach $\pm\infty$ as t increases. An example of this situation is the DE $y'' + 5y' - 6y = 0$ with solution

$$y(t) = c_1e^{t} + c_2e^{-6t}.$$

Homework: pp. 173-174, #1-17 odd, 21-25 odd, 32, 33.