Section 4.4: Nonhomogeneous Equations: The Method of Undetermined Coefficients

Judicious Guessing: Getting a Feel for the Method

In previous sections we have completely solved the problem of finding solutions to second-order (and higher) homogeneous DE's. We now begin to consider the nonhomogeneous DE

$$ay'' + by' + cy = f(t). \tag{1}$$

In this section, we restrict our attention to functions $f(t)$ which are a single term of either a polynomial, exponential, sine/cosine function or a product of these. Furthermore, we content ourselves with finding a particular solution, not the general solution.

Example 1. Find a particular solution to $y'' + 3y' + 2y = 3t$.

Solution. We want a function y such that the sum of its zeroth, first, and second derivatives is a linear function; certainly a linear function satisfies this property. If we naïvely guess a solution of the exact same form as $f(t) = 3t$, namely $y = At$, then $y' = A, y'' = 0$ implies

$$y'' + 3y' + 2y = 3A + 2At = 3t,$$

but for this equation to hold we'd need $A = 0$ (for constant terms to match) and $A = 3/2$ (for linear terms to match). Instead, try a more general linear function: $y = At + B, y' = A, y'' = 0$ implies

$$y'' + 3y' + 2y = (3A + 2B) + 2At = 3t.$$

Now we solve the system

$$2A = 3, \tag{2}$$
$$3A + 2B = 0 \tag{3}$$

to get $A = 3/2, B = -9/4$. Therefore, a solution is $y(t) = \frac{3}{2}t - \frac{9}{4}$. ◊

Generalizing this example, to find a solution to an equation of the form

$$ay'' + by' + cy = Ct^m, \tag{4}$$

we should guess the form

$$y(t) = A_m t^m + ... + A_1 t + A_0,$$

substitute this into (4) and equate coefficients of powers of t (which is solving a system of $m + 1$ equations in $m + 1$ unknowns). As the previous example demonstrated, we must retain all powers of t even if they do not appear in the nonhomogeneous term $f(t)$. This process is called the **method of undetermined coefficients**.

What if the nonhomogeneity is an exponential function e^{rt}? Since all derivatives of exponential functions remain exponential, we should try $y = Ae^{rt}$.

Example 2. Find a solution to $y'' + 3y' + 2y = 10e^{3t}$.

Solution. Using $y(t) = Ae^{3t}, y'(t) = 3Ae^{3t}, y''(t) = 9Ae^{3t}$, we have

$$y'' + 3y' + 2y = (9A + 9A + 2A)e^{3t} = 20Ae^{3t} = 10e^{3t},$$

so we must have $20A = 10 \Rightarrow A = 1/2$. Thus, a solution is $y(t) = \frac{1}{2}e^{3t}$. ◊

If the right-hand side of (1) is a sine or cosine function, its derivatives will involve both sine and cosine, so we need to guess a solution with *both* factors. So for an equation of the form

$$ay'' + by' + cy = C\sin(\beta t) \quad \text{or} \quad C\cos(\beta t),$$

we guess the solution

$$y(t) = A\cos(\beta t) + B\sin(\beta t).$$

Example 3. Find a solution to $y'' + 3y' + 2y = \sin t$.

Solution. We use $y(t) = A\sin t + B\cos t, y'(t) = A\cos t - B\sin t, y'' = -A\sin t - B\cos t$ and substitute:

$$\begin{aligned} y'' + 3y' + 2y &= (-A\sin t - B\cos t) + 3(A\cos t - B\sin t) + 2(A\sin t + B\cos t) \\ &= (A - 3B)\sin t + (3A + B)\cos t = \sin t \end{aligned}$$

This yields the system

$$A - 3B = 1, \tag{5}$$

$$3A + B = 0 \tag{6}$$

Substituting $B = -3A$ from (6) into (5) gives $10A = 1 \Rightarrow A = 1/10, B = -3/10$. Therefore, a solution is $y(t) = \frac{1}{10}\sin t - \frac{3}{10}\cos t$. ◊

If we encounter a product of such terms, we can take the product of the guesses described earlier.

Example 4. Find a solution to $y'' + 4y = 5t^2e^t$.

Solution. The nonhomogeneous term is a product of a quadratic with an exponential, so let's guess $y(t) = (At^2 + Bt + C)e^t$. Then

$$\begin{aligned} y'(t) &= (2At + B)e^t + (At^2 + Bt + C)e^t, \\ y''(t) &= 2Ae^t + 2(2At + B)e^t + (At^2 + Bt + C)e^t. \end{aligned}$$

Substituting these into the DE, we get

$$\begin{aligned} y'' + 4y &= 2Ae^t + (4At + 2B)e^t + (At^2 + Bt + C)e^t + (4At^2 + 4Bt + 4C)e^t \\ &= 5At^2e^t + (4A + 5B)te^t + (2A + 2B + 5C)e^t = 5t^2e^t. \end{aligned}$$

This yields the system of equations

$$5A = 5, \tag{7}$$

$$4A + 5B = 0, \tag{8}$$

$$2A + 2B + 5C = 0. \tag{9}$$

Then $A = 1, B = -4/5, C = -2/25$, so we have the solution

$$y(t) = \left(t^2 - \frac{4}{5}t - \frac{2}{25}\right)e^t. \qquad \diamondsuit$$

Trouble from Homogeneous Roots (and How to Overcome It)

There are some problems that our method as described so far fails to solve. For example, consider the easy-looking DE

$$y'' + y' = 5 \tag{10}$$

Since the RHS is a polynomial of degree 0, our method suggests guessing $y = A$. However, all the derivatives of this function are 0, so substituting into (10) gives $0 = 5$, a statement which is obviously false.

Likewise, for the DE

$$y'' - 6y' + 9y = e^{3t}, \tag{11}$$

our method suggests guessing $y(t) = Ae^{3t}, y'(t) = 3Ae^{3t}, y''(t) = 9Ae^{3t}$. But substituting this into (11) gives

$$(9A - 18A + 9A)e^{3t} = 0 = e^{3t},$$

again a false statement. Why is this happening? Because the solution proposed by our method is actually a solution to the corresponding *homogeneous* case. We saw that $y = A$ solves the DE $y'' + y' = 0$, and likewise, $y = Ae^{3t}$ solves $y'' - 6y' + 9y = 0$.

The way to get around this is similar to what we did in Section 4.2 for repeated roots: tack on a t. To solve (10), let's try instead $y = At$: $y' = A, y'' = 0$ implies $y'' + y' = A = 5$, so a solution is $y = 5t$. However, if we try to solve (11) by guessing $y = Ate^{3t}$, that doesn't work either because 3 is actually a repeated root of the auxiliary equation $r^2 - 6r + 9 = (r-3)^2$. So for this DE we have to tack on *two* t's:

$$y(t) = At^2e^{3t}, y'(t) = 2Ate^{3t} + 3At^2e^{3t}, y''(t) = 2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t} \Rightarrow$$

$$\begin{aligned} y'' - 6y' + 9y &= (2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t}) - 6(2Ate^{3t} + 3At^2e^{3t}) + 9(At^2e^{3t}) \\ &= 2Ae^{3t} = e^{3t}, \end{aligned}$$

which implies $A = 1/2$, so a solution is $y = \frac{1}{2}t^2e^{3t}$.

We summarize below how to apply this rule for the various functions we have looked at.

Algorithm 1. To find a particular solution to the DE

$$ay'' + by' + cy = Ct^m e^{rt},$$

where m is a nonnegative integer, guess the form

$$y(t) = t^s(A_m t^m + ... + A_1 t + A_0)e^{rt}, \tag{12}$$

where we choose

(1) $s = 0$ if r is not a root of the associated auxiliary equation;
(2) $s = 1$ if r is a simple root of the associated auxiliary equation; and
(3) $s = 2$ if r is a double root of the associated auxiliary equation.

To find a particular solution to the differential equation

$$ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t}\cos(\beta t) \\ \text{or} \\ Ct^m e^{\alpha t}\sin(\beta t) \end{cases}$$

for $\beta \neq 0$, guess the form

$$y(t) = t^s(A_m t^m + ... + A_1 t + A_0)e^{\alpha t}\cos(\beta t) + t^s(B_m t^m + ... + B_1 t + B_0)e^{\alpha t}\sin(\beta t), \tag{13}$$

where we choose

(1) $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; and
(2) $s = 1$ is $\alpha + i\beta$ is a root of the associated auxiliary equation.

Example 5. Find the **form** of a particular solution to $y'' + 2y' - 3y = f(t)$, where $f(t)$ equals

$$\text{(a)}7\cos(3t) \quad \text{(b)}2te^t\sin t \quad \text{(c)}t^2\cos(\pi t) \quad \text{(d)}5e^{-3t} \quad \text{(e)}3te^t \quad \text{(f)}t^2e^t$$

Solution. The corresponding auxiliary equation is $r^2 + 2r - 3 = (r-1)(r+3) = 0$, which has roots $r_1 = 1, r_2 = -3$. Since the functions in (a)-(c) have trig factors, and the equation has no complex roots, the solution forms have $s = 0$:
(a)$y(t) = A\cos(3t) + B\sin(3t)$,
(b)$y(t) = (A_1t + A_0)e^t\cos(t) + (B_1t + B_0)e^t\sin t$,
(c)$y(t) = (A_2t^2 + A_1t + A_0)\cos(\pi t) + (B_2t^2 + B_1t + B_0)\sin(\pi t)$.
Since -3 and 1 are both simple roots, for (d)-(f) we take $s = 1$:
(d)$y(t) = Ate^{-3t}$,
(e)$y(t) = t(A_1t + A_0)e^t$,
(f)$y(t) = t(A_2t^2 + A_1t + A_0)e^t$. ◊

Example 6. Find the form of a particular solution to $y'' - 2y' + y = f(t)$, for the same set of nonhomogeneities $f(t)$ as in Example 5.

Solution. The new auxiliary equation is $r^2 - 2r + 1 = (r-1)^2 = 0$, which has the double root $r = 1$. For (a)-(d) we can take $s = 0$, so all these are the same as the previous example except (d), which is now $y(t) = Ae^{-3t}$. For the last two, we now need $s = 2$:
(e)$y(t) = t^2(A_1t + A_0)e^t$,
(f)$y(t) = t^2(A_2t^2 + A_1t + A_0)e^t$. ◊

Example 7. Find the form of a particular solution to $y'' - 2y' + 2y = 5te^t \cos t$.

Solution. The auxiliary equation is $r^2 - 2r + 2 = 0$ with roots

$$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Since the nonhomogeneity involves $e^t \cos t$ with $\alpha = \beta = 1$, we must take $s = 1$:

$$y(t) = t(A_1 t + A_0)e^t \cos t + t(B_1 t + B_0)e^t \sin t. \qquad \diamond$$

Obviously this method is incomplete, as there are several types of functions it cannot handle, and we have not found any general solutions. These deficiencies will begin to be remedied in the next section.

Homework: p. 182 #9-31 odd.