Section 4.6: Variation of Parameters

A New Method for Nonhomogeneous DE's

We have examined one procedure for finding particular solutions to nonhomogeneous DE's, namely, the method of undetermined coefficients. In this section, we present an alternate method, which has the advantage of applying to any function (as opposed to UC, which only applies for certain functions). We are trying to solve the nonhomogeneous second-order DE

$$ay'' + by' + cy = f(t). \tag{1}$$

Recall that if $y_1(t), y_2(t)$ are linearly independent solutions of the corresponding homogeneous equation $ay'' + by' + cy = 0$, then a general solution to the homogeneous equation is

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t) \tag{2}$$

for arbitrary constants c_1, c_2. Our new strategy for finding a particular solution of the nonhomogeneous DE (1) is to replace the constants in (2) with functions of t. Thus, we want a solution of the form

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t). \tag{3}$$

Of course, not just any functions v_1, v_2 will work; they will need to satisfy some conditions in order for (3) to actually be a solution. To determine those conditions, let's assume that (3) *is* a solution to (1), and see what falls out. Computing derivatives of the particular solution gives

$$y_p' = (v_1' y_1 + v_2' y_2) + (v_1 y_1' + v_2 y_2').$$

Notice that the derivative of the first term in parentheses will include second derivatives of our unknown v functions; since this would make the functions harder to calculate, let's impose the condition that the mentioned term drops out:

$$v_1' y_1 + v_2' y_2 = 0. \tag{4}$$

Then the derivative simplifies to

$$y_p' = v_1 y_1' + v_2 y_2'$$

and thus

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''.$$

Substituting these expressions into (1) gives

$$\begin{aligned} f &= a y_p'' + b y_p' + c y_p \\ &= a(v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'') + b(v_1 y_1' + v_2 y_2') + c(v_1 y_1 + v_2 y_2) \\ &= a(v_1' y_1' + v_2' y_2') + v_1(a y_1'' + b y_1' + c y_1) + v_2(a y_2'' + b y_2' + c y_2) \\ &= a(v_1' y_1' + v_2' y_2') + 0 + 0 \end{aligned}$$

since y_1 and y_2 solve the homogeneous equation. Therefore, a second condition is

$$v_1'y_1' + v_2'y_2' = \frac{f}{a}. \tag{5}$$

In conclusion, if we can find functions v_1, v_2 that solve the system of equations (4) and (5), then (3) will be a particular solution to (1). One could solve this system generally using algebraic manipulation or Cramer's rule, but the resulting formulas are complicated and difficult to remember. It is better to just solve the system of equations generated by the method.

Example 1. Find a general solution on $(-\pi/2, \pi/2)$ to $y'' + y = \tan t$.

Solution. The auxiliary equation for the homogeneous equation is $r^2 + 1 = 0$ with roots $r = \pm i$, so linearly independent solutions are $y_1 = \cos t, y_2 = \sin t$. Therefore, our particular solution has the form

$$y_p(t) = v_1(t)\cos t + v_2(t)\sin t.$$

The system of equations to solve is

$$\begin{aligned} (\cos t)v_1' + (\sin t)v_2' &= 0, && (6)\\ (-\sin t)v_1' + (\cos t)v_2' &= \tan t. && (7) \end{aligned}$$

Multiplying (6) by $\sin t$ and (7) by $\cos t$ gives

$$\begin{aligned} (\sin t\cos t)v_1' + (\sin^2 t)v_2' &= 0, && (8)\\ (-\sin t\cos t)v_1' + (\cos^2 t)v_2' &= \tan t\cos t. && (9) \end{aligned}$$

Adding them gives $v_2' = \tan t\cos t = \sin t$, so $v_2 = \int \sin t dt = -\cos t$. Solving (6) for v_1' gives

$$(\cos t)v_1' = -\sin^2 t \Rightarrow v_1' = -\frac{\sin^2 t}{\cos t}.$$

Thus, we have

$$\begin{aligned} v_1(t) &= -\int \frac{\sin^2 t}{\cos t}dt = -\int \frac{1-\cos^2 t}{\cos t}dt\\ &= \int (\cos t - \sec t)dt = \sin t - \ln|\sec t + \tan t|. \end{aligned}$$

Substituting these, we obtain the particular solution

$$y_p(t) = (\sin t - \ln|\sec t + \tan t|)\cos t - \cos t\sin t = -(\cos t)\ln(\sec t + \tan t),$$

where we have dropped the absolute value bars because the expression $\sec t + \tan t = \dfrac{1+\sin t}{\cos t} > 0$ on the given interval.

Finally, a general solution is given by adding the particular solution to the general solution of the homogeneous equation:

$$y(t) = c_1\cos t + c_2\sin t - (\cos t)\ln(\sec t + \tan t). \qquad \Diamond$$

Example 2. Find a particular solution on $(-\pi/2, \pi/2)$ to $y'' + y = \tan t + 3t - 1$.

Solution. We could apply the variation of parameters to $f(t) = \tan t + 3t - 1$. But it will be easier to solve the following equations separately, and then apply the superposition principle:

$$y'' + y = \tan t, \tag{10}$$

$$y'' + y = 3t - 1. \tag{11}$$

In Example 1, we found the particular solution $y_p(t) = -(\cos t)\ln(\sec t + \tan t)$ to equation (10). For equation (11), we can use undetermined coefficients. We guess $y_p(t) = At + B, y_p' = A, y_p'' = 0$, so that $y_p'' + y_p = At + B = 3t - 1$, so a particular solution to (11) is $y_p(t) = 3t - 1$. According to the superposition principle, a particular solution to the original DE is given by their sum:

$$y(t) = -(\cos t)\ln(\sec t + \tan t) + 3t - 1. \qquad \Diamond$$

Example 3. Find a general solution to $y'' - 2y' + y = t^{-1}e^t$.

Solution. The associated homogeneous equation has auxiliary equation $r^2 - 2r + 1 = (r-1)^2 = 0$, so $r = 1$ is a double root, and a general solution is given by

$$y_h(t) = c_1e^t + c_2te^t.$$

To find a particular solution, we let $y_p(t) = v_1(t)e^t + v_2(t)te^t$. We need to solve the system

$$e^tv_1' + te^tv_2' = 0, \tag{12}$$

$$e^tv_1' + (e^t + te^t)v_2' = t^{-1}e^t. \tag{13}$$

Subtracting (12) from (13) gives

$$e^tv_2' = t^{-1}e^t \Rightarrow v_2' = \frac{1}{t} \Rightarrow v_2(t) = \int \frac{dt}{t} = \ln|t|.$$

Substituting $v_2' = t^{-1}$ into (12) gives

$$e^tv_1' = -e^t \Rightarrow v_1' = -1 \Rightarrow v_1 = -\int 1dt = -t.$$

Therefore, a particular solution is

$$y_p(t) = -te^t + te^t\ln|t|$$

and a general solution is given by

$$y(t) = y_h(t) + y_p(t) = c_1e^t + c_2te^t + te^t\ln|t|. \qquad \Diamond$$

Homework: p. 193 #1-17 odd.