## Section 4.7: Variable-Coefficient Equations

### Cauchy-Euler Equations

Before concluding our study of second-order linear DE's, let us summarize what we've done. In Sections 4.2 and 4.3 we showed how to find general solutions to homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0$$

using the auxiliary equation. (We also extended those techniques to higher-order DE's in Section 6.2.) In Sections 4.4-4.6 we looked at two different methods, namely, undetermined coefficients and variation of parameters, for solving nonhomogeneous equations, still with constant coefficients:

$$ay'' + by' + cy = f(t).$$

The final level of complexity, which we discuss in this section, involves nonhomogeneous equations with *variable* coefficients:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t), \tag{1}$$

or dividing through by the leading coefficient, we have the **standard form**

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t). \tag{2}$$

The major difficulty with such equations is that in general we cannot construct explicit solutions. Therefore, before describing the more general theory, we mention a special class of equations which *can* be solved explicitly.

**Definition.** A linear second-order DE that can be expressed in the form

$$at^2y''(t) + bty'(t) + cy(t) = f(t), \tag{3}$$

for constants $a, b, c \in \mathbb{R}$ is called a Cauchy-Euler (or equidimensional) equation.

The terminology "equidimensional" arises from the fact that each of the terms $t^2y'', ty'$, and $y$ have the same dimensions (think of $y$ in meters and $t$ in seconds, for example). Observe that because the standard form

$$y'' + \frac{b}{at}y' + \frac{c}{at^2}y = \frac{f(t)}{at^2}$$

is discontinuous at $t = 0$, we expect that solutions will be valid only for $t > 0$ or $t < 0$.

To solve the associated homogeneous Cauchy-Euler equation, we utilize the equidimensionality by guessing a solution of the form $y = t^r$. This way, each of the terms mentioned are a constant multiple of $t^r$:

$$y = t^r, \quad ty' = trt^{r-1} = rt^r, \quad t^2y'' = t^2r(r-1)t^{r-2} = r(r-1)t^r.$$

If we substitute these into the homogeneous form of (3) (with $f = 0$), we get a quadratic equation:

$$ar(r-1)t^r + brt^r + ct^r = [ar^2 + (b-a)r + c]t^r = 0 \Rightarrow$$

$$ar^2 + (b-a)r + c = 0. \tag{4}$$

To distinguish it from our earlier discussions, we call (4) the **characteristic equation**.

**Example 1.** Find two linearly independent solutions to the equation

$$3t^2y'' + 11ty' - 3y = 0, \qquad t > 0.$$

**Solution.** The characteristic equation is

$$3r^2 + 8r - 3 = (3r-1)(r+3) = 0,$$

so there are distinct roots $r_1 = 1/3, r_2 = -3$. Since our solutions are of the form $y = t^r$, two linearly independent solutions are $y_1(t) = t^{1/3}, y_2(t) = t^{-3}$. ◊

The analysis in this situation is nearly identical to that encountered in Sections 4.2 and 4.3. Let us discuss the possibilities of complex and repeated roots. If the characteristic equation has the complex root $r = \alpha + i\beta$, we use the identity $t = e^{\ln t}$ and Euler's formula to write

$$t^{\alpha+i\beta} = t^\alpha t^{i\beta} = t^\alpha e^{i\beta \ln t} = t^\alpha[\cos(\beta \ln t) + i \sin(\beta \ln t)].$$

As before, the corresponding linearly independent solutions are the real and imaginary parts:

$$y_1 = t^\alpha \cos(\beta \ln t), y_2 = t^\alpha \sin(\beta \ln t).$$

If $r$ is a double root of the characteristic equation, then linearly independent solutions are

$$y_1 = t^r, y_2 = t^r \ln t.$$

**Example 2.** Find a pair of linearly independent solutions to the Cauchy-Euler equations for $t > 0$: (a) $t^2y'' + 5ty' + 5y = 0$ (b) $t^2y'' + ty' = 0$

**Solution.** (a) The characteristic equation is $r^2 + 4r + 5 = 0$ with roots

$$r = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i,$$

so linearly independent solutions are $y_1 = t^{-2}\cos(\ln t), y_2 = t^{-2}\sin(\ln t)$.
(b) The characteristic equation is $r^2 = 0$ with double root $r = 0$, so linearly independent solutions are $y_1 = t^0 = 1, y_2 = \ln t$. ◊

*Remark.* If we are solving a homogeneous Cauchy-Euler equation for $t < 0$, one simply makes the substitution $t = -\tau$ for $\tau > 0$. This results in the same characteristic equation, so the only difference is that in the solutions $t$ is replaced by $-t$.

## Extension of Previous Theory

Recall that the main difficulty of variable-coefficient equations is that we cannot find explicit general solutions. However, assuming that we are given explicit solutions, we show that many of our previous results are still valid. We begin by giving an alternate characterization of linear independence, and then show that general solutions for both homogeneous and nonhomogeneous equations are found in the same way as before.

**Definition.** Let $y_1$ and $y_2$ be differentiable functions. We define the Wronskian of $y_1$ and $y_2$ as

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

**Lemma 1.** *Let $p(t)$ and $q(t)$ be continuous on the interval $I$ and suppose $y_1(t)$ and $y_2(t)$ are solutions on $I$ to the homogeneous DE*

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0. \tag{5}$$

*If the Wronskian $W[y_1, y_2](t) \neq 0$ at any point in $I$, then $y_1$ and $y_2$ are linearly independent on $I$.*

**Theorem 2.** *If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous DE (5) which are linearly independent on the interval $I$ (that is, $W[y_1, y_2] \neq 0$ for some point in $I$), then a general solution to (5) on $I$ is $y_h = c_1y_1 + c_2y_2$ for arbitrary constants $c_1, c_2 \in \mathbb{R}$.*

The superposition principle can be extended to variable-coefficient equations, which implies the following result.

**Theorem 3.** *A general solution for the nonhomogeneous equation (2) on $I$ is given by*

$$y = y_p + y_h,$$

*where $y_h$ is a general solution to the corresponding homogeneous equation (5) on $I$ and $y_p$ is a particular solution to (2) on $I$.*

If we are given linearly independent solutions to equation (5), then the method of variation of parameters can be used to determine a particular solution for the nonhomogeneous equation (2). In fact, the definition of the Wronskian even allows us to give a formula for the functions $v_1$ and $v_2$. (Unfortunately, the method of undetermined coefficients does not extend to variable-coefficient equations.)

**Theorem 4.** *If $y_1$ and $y_2$ are two linearly independent solutions to the homogeneous equation (5) on an interval $I$ where $p(t), q(t)$, and $g(t)$ are continuous, then a particular solution to (2) is given by $y_p = v_1y_1 + v_2y_2$, where $v_1, v_2$ are determined by the system*

$$\begin{aligned} y_1v_1' + y_2v_2' &= 0, \\ y_1'v_1' + y_2'v_2' &= g, \end{aligned}$$

*which has the solution*

$$v_1(t) = \int \frac{-g(t)y_2(t)}{W[y_1, y_2](t)}dt, \qquad v_2(t) = \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)}dt.$$

The conclusion is that if we are handed two linearly independent solutions to the homogeneous equation (5) at the outset, we can proceed as before to find a general solution to (2) as well as solve initial value problems. But (in general) we cannot produce such linearly independent solutions on our own.

**Reduction of Order**

Actually, what we have just said is not quite true. It turns out that if we are given just one nontrivial solution to (5), there is a way to construct a second, linearly independent solution, so that we can proceed from that point as outlined above.

**Theorem 5.** *(Reduction of Order) Let $y_1(t)$ be a solution (not identically zero) to the homogeneous DE (5) on an interval $I$. Then a second, linearly independent solution is given by*

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{y_1(t)^2} dt \tag{6}$$

Sometimes this formula can be applied directly, but other times it will be easier to follow the logic of the proof, so we give an outline of the proof below.

*Proof.* We proceed as in Section 4.6 when constructing the variation of parameters method. Since $cy_1$ is a solution of (5) for any constant, try a solution of the form $y_2(t) = v(t)y_1(t)$. Then

$$y_2' = vy_1' + v'y_1, \qquad y_2'' = vy_1'' + 2v'y_1' + v''y_1.$$

Substituting these into (5) yields

$$\begin{aligned} (vy_1'' + 2v'y_1' + v''y_1) + p(vy_1' + v'y_1) + qvy_1 &= 0, \\ (y_1'' + py_1' + qy_1)v + y_1v'' + (2y_1' + py_1)v' &= 0, \\ y_1v'' + (2y_1' + py_1)v' &= 0, \end{aligned}$$

where the first term of the middle equation drops out because $y_1$ is a solution to (5). Observe that the final equation is actually a separable first-order equation in $w := v'$, so it can easily be solved. The remainder of the proof is left to the overly enthusiastic reader. □

**Example 3.** Given that $y_1(t) = t$ is a solution to $y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0$, use reduction of order to find a second linearly independent solution for $t > 0$.

**Solution.** We see that $p(t) = -\frac{1}{t}$, so $e^{-\int p(t)dt} = e^{\int 1/tdt} = e^{\ln t} = t$. With $y_1(t) = t$ in (6), we get

$$y_2(t) = t \int \frac{t}{t^2} dt = t \int \frac{1}{t} dt = t \ln t.$$

Alternatively, multiplying through by $t^2$ gives a Cauchy-Euler equation with characteristic equation $r^2 - 2r + 1 = (r-1)^2 = 0$ having $r = 1$ as a double root, which gives the solutions mentioned. ◊

**Example 4.** Find a general solution to the following DE which comes from modeling reverse osmosis:

$$(\sin t)y'' - 2(\cos t)y' - (\sin t)y = 0, \qquad 0 < t < \pi.$$

**Solution.** The first challenge is to find one explicit solution to the DE. Since there are trig functions involved, one might hope that either $\sin t$ or $\cos t$ works; the first fails (you may want to verify this for yourself), but the second works:

$$y_1 = \cos t, \qquad y_1' = -\sin t \qquad y_1'' = -\cos t \Rightarrow$$
$$(\sin t)y_1'' - 2(\cos t)y_1' - (\sin t)y_1 = -\sin t\cos t + 2\sin t\cos t - \sin t\cos t = 0.$$

Now we can use reduction of order to find a second solution. If we try to use the formula in (6), we'll have to integrate an exponential with a trig exponent divided by a trig function, which sounds unpleasant to say the least. Instead, let's follow the proof. Let $y_2(t) = v(t)y_1(t) = v(t)\cos t$; then $y_2' = v'\cos t - v\sin t$ and $y_2'' = v''\cos t - 2v'\sin t - v\cos t$. Substituting this into the original DE gives

$$(\sin t)[v''\cos t - 2v'\sin t - v\cos t] - 2(\cos t)[v'\cos t - v\sin t] - (\sin t)(v\cos t)$$
$$= v''(\sin t\cos t) - 2v'(\sin^2 t + \cos^2 t) + v(-\sin t\cos t + 2\sin t\cos t - \sin t\cos t) = 0 \Rightarrow$$
$$v''(\sin t\cos t) - 2v' = 0.$$

Separating this equation gives

$$\frac{(v')'}{(v')} = \frac{2}{\sin t\cos t} = 2\frac{\sec^2 t}{\tan t},$$

so integrating (with a $u$-substitution) yields $\ln v' = 2\ln(\tan t) \Rightarrow v' = \tan^2 t$, and integrating again, we have $v(t) = \tan t - t$ (we have used the trig identity $\tan^2 t = \sec^2 t - 1$). Therefore, the second solution is $y_2(t) = (\tan t - t)\cos t = \sin t - t\cos t$ and the general solution is

$$y(t) = c_1\cos t + c_2(\sin t - t\cos t). \qquad \Diamond$$

Homework: pp. 200-202 #9-19 odd, 37-47 odd.