Section 6.1: Basic Theory of Higher-Order Linear Differential Equations

General Solutions and Linear Independence for Higher-Order DE's

In this section, we conclude Unit 2 by looking at how the theorems we had about general solutions to second-order linear DE's extend also to higher-order DE's. We shall present the results largely without rigorous proof; the reader who is interested in details may consult the text.

Recall that a linear differential equation of order n has the form

$$(1) \qquad a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + ... + a_0(x)y(x) = b(x),$$

where $a_n(x) \neq 0$ and all functions are continuous on an interval I. If $a_0, a_1, ..., a_n$ are all constants, we say equation (1) has **constant coefficients**; if not, it has **variable coefficients**. If $b(x) = 0$, equation (1) is called **homogeneous**; otherwise it is **nonhomogeneous**. The **standard form** of equation (1) is

$$(2) \qquad y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + ... + p_n(x)y(x) = g(x),$$

where again all functions are continuous on I.

The first result we consider involves the general solution of a homogeneous equation. Recall that for second-order DE's, these were linear combinations of two linearly independent solutions. One might guess (correctly) that the general solution for a homogeneous DE of order n is a linear combination of n linearly independent solutions. Therefore, we need to generalize our earlier definition of linear independence to accommodate any number of functions.

Definition. The m functions $f_1, f_2, \dots, f_m$ are said to be linearly dependent on an interval I if there exist constants $c_1, c_2, \dots, c_m$ not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0$$

for all $x \in I$. Otherwise, they are said to be linearly independent on I.

Some common linearly independent sets are

(1) $\{1, x, x^2, \dots, x^n\}$

(2) $\{1, \cos x, \sin x, \cos(2x), \sin(2x), ..., \cos(nx), \sin(nx)\}$

(3) $\{e^{\alpha_1 x}, e^{\alpha_2 x}, ..., e^{\alpha_n x}\}$ where α_i's are distinct constants

It is not hard to see that determining linear independence can become increasingly more complex as the number of functions increases. To remedy this, we generalize the Wronskian function introduced in Section 4.7.

Definition. Let $f_1, ..., f_n$ be any n functions that are $(n-1)$ times differentiable. The function

$$W[f_1, ..., f_n](x) := \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix} \tag{3}$$

is called the <u>Wronskian</u> of $f_1, f_2, ..., f_n$.

It turns out that the functions $y_1, y_2, ..., y_n$ are linearly independent on the interval (a, b) if and only if the Wronskian $W[y_1, y_2, ..., y_n](x_0) \neq 0$ for some $x_0 \in (a, b)$. We can now formally state the result we intuitively guessed.

Theorem 1. *Let $y_1, ..., y_n$ be n solutions on (a, b) of the homogeneous DE*

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0, \tag{4}$$

where $p_1, ..., p_n$ are continuous on (a, b). If at some point $x_0 \in (a, b)$ the solutions satisfy $W[y_1, y_2, ..., y_n](x_0) \neq 0$, then a general solution for (4) is

$$y(x) = C_1y_1(x) + \cdots + C_ny_n(x), \tag{5}$$

*where $C_1, ..., C_n$ are arbitrary constants. The set $\{y_1, y_2, \ldots, y_n\}$ is called a **fundamental solution set** for (4) on (a, b).*

What about the nonhomogeneous case? Again, it falls out exactly how we might hope it would.

Theorem 2. *Let $y_p(x)$ be a particular solution on the interval (a, b) to the nonhomogeneous equation*

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x), \tag{6}$$

where $p_1, p_2, \ldots, p_n$ are continuous on (a, b), and let $\{y_1, ..., y_n\}$ be a fundamental solution set for the corresponding homogeneous equation (4). Then a general solution for (6) on the interval (a, b) is

$$y(x) = y_p(x) + C_1y_1(x) + \cdots + C_ny_n(x). \tag{7}$$

A final helpful result is an extension of the superposition principle.

Theorem 3. *(Generalized Superposition Principle) Let y_{p_1} be a particular solution to the DE*

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g_1(x),$$

let y_{p_2} be a particular solution to the DE

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g_2(x),$$

and let $\{y_1, ...y_n\}$ be a fundamental solution set for the corresponding homogeneous DE (4). Then for any constants c_1, c_2, a general solution to the DE

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = c_1g_1(x) + c_2g_2(x)$$

is given by

$$y(x) = c_1 y_{p_1}(x) + c_2 y_{p_2}(x) + C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x),$$

for arbitrary constants $C_1, C_2, ..., C_n$.

Example 1. Find a general solution on the interval $(-\infty, \infty)$ to

$$y''' - 2y'' - y' + 2y = 2x^2 - 2x - 4 - 24e^{-2x},$$

given that $y_{p_1}(x) = x^2$ is a particular solution to $y''' - 2y'' - y' + 2y = 2x^2 - 2x - 4$, $y_{p_2}(x) = e^{-2x}$ is a particular solution to $y''' - 2y'' - y' + 2y = -12e^{-2x}$, and that $y_1(x) = e^{-x}, y_2(x) = e^x$, and $y_3(x) = e^{2x}$ are solutions to the corresponding homogeneous equation.

Solution. Observe that the three solutions to the homogeneous equation are linearly independent because the exponents are all distinct. Therefore, $\{e^{-x}, e^x, e^{2x}\}$ is a fundamental solution set. Since the right-hand side of our DE is $g_1(x) + 2g_2(x)$ for $g_1(x) = 2x^2 - 2x - 4$ and $g_2(x) = -12e^{-2x}$, by the generalized superposition principle we have the general solution

$$\begin{aligned} y(x) &= y_{p_1} + 2y_{p_2} + C_1 y_1 + C_2 y_2 + C_3 y_3 \\ &= x^2 + 2e^{-2x} + C_1 e^{-x} + C_2 e^x + C_3 e^{2x}. \quad \diamondsuit \end{aligned}$$

Homework: p. 325 #7-13 odd, 19-23 odd.