Section 6.2: Higher-Order Homogeneous Linear Equations with Constant Coefficients

Introduction: The Auxiliary Equation

Now that we understand how to solve second-order homogeneous linear DE's with constant coefficients, we seek to generalize these concepts to write solutions for nth-order homogeneous linear DE's with constant coefficients. Recall that such a DE has the form

$$a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + ... + a_1 y'(x) + a_0 y(x) = 0, \tag{1}$$

where $a_n \neq 0$ and $a_i \in \mathbb{R}$ for all i. Remember that when we found two linearly independent solutions to the second-order DE, we could write the general solution as a linear combination of these solutions. Likewise, if $y_1(x), y_2(x), ..., y_n(x)$ are linearly independent solutions to (1), then the general solution of (1) has the form

$$y(x) = c_1 y_1(x) + ... + c_n y_n(x) \tag{2}$$

How can we find these linearly independent solutions? Just as with second-order DE's, we try a function of the form $y = e^{rx}$. Observing that $\dfrac{d^k}{dx^k}(e^{rx}) = r^k e^{rx}$, substituting $y = e^{rx}$ into the left-hand side of equation (1) gives

$$\begin{aligned} a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) ... + a_0 y(x) &= a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + ... + a_0 e^{rx} \\ &= e^{rx}(a_n r^n + a_{n-1} r^{n-1} + ... + a_0) = e^{rx} P(r). \end{aligned}$$

Therefore, e^{rx} will be a solution to (1) exactly when r is a root of the auxiliary equation

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0 = 0. \tag{3}$$

The fundamental theorem of algebra tells us that the auxiliary equation will have n roots counting multiplicity, which can be real or complex. Although we could use the quadratic formula for the second-order case, for higher order equations we will need to test roots and factor polynomials using synthetic division, just as in a precalculus course. It is recommended that one first check whether 0 or ± 1 are roots as these are relatively easy to see, before proceeding to use synthetic division to check for other roots. We now examine the terms which occur in the general solution depending on the nature of the roots of the auxiliary equation.

Distinct Real Roots

If the roots $r_1, ..., r_n$ of the auxiliary equation (3) are all real and distinct, then similar to the second-order case, we have the linearly independent solutions

$$y_1(x) = e^{r_1 x}, y_2(x) = e^{r_2 x}, ..., y_n(x) = e^{r_n x}.$$

(Recall that these are linearly independent since the coefficients of x in the exponent are all distinct.) Therefore, in this case a general solution to (1) is

$$y(x) = c_1 e^{r_1 x} + ... + c_n e^{r_n x}$$

for arbitrary constants $c_1, ..., c_n$.

Example 1. Find a general solution to $y''' - 2y'' - 5y' + 6y = 0$.

Solution. The auxiliary equation is $r^3 - 2r^2 - 5r + 6 = 0$. Since the sum of the coefficients is zero, $r = 1$ is a root. Using synthetic division, we can then factor the auxiliary equation as

$$r^3 - 2r^2 - 5r + 6 = (r-1)(r^2 - r - 6) = (r-1)(r-3)(r+2) = 0.$$

Therefore, the auxiliary equation has three distinct real roots $r_1 = 1, r_2 = 3, r_3 = -2$, and the general solution is

$$y(x) = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}. \qquad \diamond$$

Complex Roots

If $\alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$ is a complex root of the auxiliary equation (3), then so is its complex conjugate $\alpha - i\beta$ (this is because the coefficients of $P(r)$ are real). Then as described in Section 4.3, $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ are complex-valued solutions to (1). To get real-valued solutions, we again apply Euler's formula to write $e^{(\alpha+i\beta)x} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x)$; then two linearly independent solutions to (1) are $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.

Example 2. Find a general solution to $y''' + y'' + 3y' - 5y = 0$.

Solution. The auxiliary equation is $r^3 + r^2 + 3r - 5 = 0$. Again, the sum of the coefficients is zero, so $r = 1$ is a root. By synthetic division,

$$r^3 + r^2 + 3r - 5 = (r-1)(r^2 + 2r + 5) = 0.$$

To find the roots of the quadratic factor, apply the quadratic formula:

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i,$$

so in this case $\alpha = -1, \beta = 2$. Therefore, a general solution is

$$y(x) = c_1 e^x + c_2 e^{-x}\cos(2x) + c_3 e^{-x}\sin(2x). \qquad \diamond$$

Repeated Roots

Now suppose that r_1 is a real root of equation (3) with multiplicity m. Recall that for a second-order DE, a root r of multiplicity 2 produced the linearly independent solutions e^{rx} and xe^{rx}. Likewise, we get m linearly independent solutions to (1) from the single repeated root r_1 as follows:

$$e^{r_1 x}, xe^{r_1 x}, x^2 e^{r_1 x}, ..., x^{m-1} e^{r_1 x}.$$

(A formal proof that these functions are truly solutions to (1) is given on pp. 329-330 in the text.)

If $\alpha + i\beta$ is a complex root of equation (3) with multiplicity m (and therefore its conjugate $\alpha - i\beta$ is also a root of multiplicity m), the expected happens: we get $2m$ complex-valued solutions

$$e^{(\alpha+i\beta)x}, xe^{(\alpha+i\beta)x}, ..., x^{m-1}e^{(\alpha+i\beta)x},$$
$$e^{(\alpha-i\beta)x}, xe^{(\alpha-i\beta)x}, ..., x^{m-1}e^{(\alpha-i\beta)x}$$

which can be replaced by the $2m$ linearly independent real-valued solutions

$$e^{\alpha x}\cos(\beta x), xe^{\alpha x}\cos(\beta x), ..., x^{m-1}e^{\alpha x}\cos(\beta x),$$
$$e^{\alpha x}\sin(\beta x), xe^{\alpha x}\sin(\beta x), ..., x^{m-1}e^{\alpha x}\sin(\beta x).$$

Example 3. Find a general solution to $y^{(4)} - y^{(3)} - 3y'' + 5y' - 2y = 0$.

Solution. The auxiliary equation is $r^4 - r^3 - 3r^2 + 5r - 2 = 0$. Since the sum of the coefficients is zero, $r = 1$ is a root, and by synthetic division,

$$r^4 - r^3 - 3r^2 + 5r - 2 = (r-1)(r^3 - 3r + 2).$$

Observe that the cubic factor has coefficients which also sum to zero, so $r = 1$ is a root of this polynomial as well; using synthetic division again, we get

$$r^4 - r^3 - 3r^2 + 5r - 2 = (r-1)^2(r^2 + r - 2) = (r-1)^3(r+2) = 0.$$

Therefore, $r_1 = 1$ is a root of multiplicity 3, and $r_2 = -2$ is a root of multiplicity 1. Hence, a general solution is

$$y(x) = c_1e^x + c_2xe^x + c_3x^2e^x + c_4e^{-2x}. \qquad \Diamond$$

Example 4. Find a general solution to $y^{(4)} - 8y^{(3)} + 26y'' - 40y' + 25y = 0$, whose auxiliary equation can be factored as

$$r^4 - 8r^3 + 26r^2 - 40r + 25 = (r^2 - 4r + 5)^2 = 0.$$

Solution. We find the roots of the repeated quadratic factor using the quadratic formula:

$$r = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

Thus, $\alpha = 2, \beta = 1$, and these complex roots each have multiplicity 2 since this factor is squared in the auxiliary equation. Therefore, a general solution is

$$y(x) = c_1e^{2x}\cos x + c_2xe^{2x}\cos x + c_3e^{2x}\sin x + c_4xe^{2x}\sin x. \qquad \Diamond$$

Homework: p. 331 #1-13 odd, 19, 21.