Section 7.2: Definition of the Laplace Transform

Basic Laplace Transforms

We begin Unit 3 by introducing the notion of a Laplace transform. This may seem strange at the outset, but before long we will see its usefulness in solving differential equations. The Laplace transform is known as an "integral operator"; it involves integration, and the word "operator" signals that it takes a function as its input and produces a new function.

Definition. Let $f(t)$ be a function on $[0,\infty)$. The Laplace transform of f is the function F defined by the integral

$$F(s) := \int_0^\infty e^{-st} f(t)dt. \tag{1}$$

The domain of $F(s)$ is all values of s for which the integral (1) exists. The Laplace transform is denoted by either F or $\mathscr{L}\{f\}$.

Recall that the integral in (1) is an improper integral, and is calculated by evaluating the limit of a definite integral:

$$\int_0^\infty e^{-st} f(t)dt = \lim_{N\to\infty} \int_0^N e^{-st} f(t)dt.$$

We now gain familiarity with this definition by finding the Laplace transforms of some basic functions.

Example 1. Determine the Laplace transform of the constant function $f(t) = 1, t \geq 0$.

Solution. By the definition, we compute

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{N\to\infty} \int_0^N e^{-st} dt \\ &= \lim_{N\to\infty} \frac{-e^{-st}}{s}\bigg|_{t=0}^{t=N} = \lim_{N\to\infty} \left[\frac{1}{s} - \frac{e^{-sN}}{s}\right]. \end{aligned}$$

If $s > 0$ is fixed, then $e^{-sN} \to 0$ as $N \to \infty$, so we have $F(s) = \dfrac{1}{s}$ for $s > 0$. If $s \leq 0$, the integral diverges since $e^{-sN} \to \infty$. ◊

Example 2. Determine the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution. By the definition, we have

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt \\ &= \lim_{N\to\infty} \int_0^N e^{-(s-a)t} dt = \lim_{N\to\infty} \frac{-e^{-(s-a)t}}{s-a}\bigg|_0^N \\ &= \lim_{N\to\infty} \left[\frac{1}{s-a} - \frac{e^{-(s-a)N}}{s-a}\right] = \frac{1}{s-a} \end{aligned}$$

for $s > a$. As before, if $s \leq a$ the integral diverges. Note that our solution agrees with that of Example 1 when $a = 0$. $\Diamond$

Example 3. Find $\mathscr{L}\{\sin(bt)\}$, where $b \neq 0$ is a constant.

Solution. We need to find $\mathscr{L}\{\sin(bt)\}(s) = \int_0^\infty e^{-st}\sin(bt)dt = \lim_{N\to\infty}\int_0^N e^{-st}\sin(bt)dt$. Through a "wraparound" integration by parts, we get

$$\begin{aligned}\mathscr{L}\{\sin(bt)\}(s) &= \lim_{N\to\infty}\left[\frac{e^{-st}}{s^2+b^2}(-s\sin(bt)-b\cos(bt))\Big|_0^N\right]\\ &= \lim_{N\to\infty}\left[\frac{b}{s^2+b^2}-\frac{e^{-sN}}{s^2+b^2}(s\sin(bN)+b\cos(bN))\right]\\ &= \frac{b}{s^2+b^2}\end{aligned}$$

for $s > 0$, since it is only for those values that $\lim_{N\to\infty} e^{-sN}(s\sin(bN)+b\cos(bN)) = 0$. $\Diamond$

Example 4. Determine the Laplace transform of

$$f(t) = \begin{cases} 2, & 0 < t < 5,\\ 0, & 5 < t < 10,\\ e^{4t}, & t > 10.\end{cases}$$

Solution. Since our function has a different formula on three different intervals, we'll need to break up the integral into three parts:

$$\begin{aligned}F(s) &= \int_0^5 e^{-st}\cdot 2dt + \int_5^{10} e^{-st}\cdot 0dt + \int_{10}^\infty e^{-st}e^{4t}dt\\ &= 2\int_0^5 e^{-st}dt + \lim_{N\to\infty}\int_{10}^N e^{-(s-4)t}dt\\ &= 2\left[\frac{e^{-st}}{-s}\Big|_0^5\right] + \lim_{N\to\infty}\left[\frac{e^{-10(s-4)}}{s-4}-\frac{e^{-(s-4)N}}{s-4}\right]\\ &= \frac{2}{s}-\frac{2e^{-5s}}{s}+\frac{e^{-10(s-4)}}{s-4},\end{aligned}$$

which is defined for $s > 4$. $\Diamond$

Linearity of the Laplace Transform

We shall discuss several properties of the Laplace transform in the next section, but here mention the crucial property of linearity.

Theorem 1. *Let f, f_1, f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then for $s > \alpha$,*

$$\mathscr{L}\{f_1 + f_2\} = \mathscr{L}\{f_1\} + \mathscr{L}\{f_2\}, \tag{2}$$

$$\mathscr{L}\{cf\} = c\mathscr{L}\{f\}. \tag{3}$$

Proof. These properties fall out simply from the linearity of integration:

$$\begin{aligned} \mathscr{L}\{f_1 + f_2\}(s) &= \int_0^\infty e^{-st}[f_1(t) + f_2(t)]dt \\ &= \int_0^\infty e^{-st} f_1(t)dt + \int_0^\infty e^{-st} f_2(t)dt \\ &= \mathscr{L}\{f_1\}(s) + \mathscr{L}\{f_2\}(s). \end{aligned}$$

Similarly, we have

$$\mathscr{L}\{cf\}(s) = \int_0^\infty e^{-st}[cf(t)]dt = c\int_0^\infty e^{-st} f(t)dt = c\mathscr{L}\{f\}(s). \qquad \square$$

Example 5. Determine $\mathscr{L}\{11 + 5e^{4t} - 6\sin(2t)\}$.

Solution. By the linearity property, we know that

$$\begin{aligned} \mathscr{L}\{11 + 5e^{4t} - 6\sin(2t)\} &= \mathscr{L}\{11\} + \mathscr{L}\{5e^{4t}\} + \mathscr{L}\{-6\sin(2t)\} \\ &= 11\mathscr{L}\{1\} + 5\mathscr{L}\{e^{4t}\} - 6\mathscr{L}\{\sin(2t)\}. \end{aligned}$$

In the first three examples, we calculated

$$\mathscr{L}\{1\}(s) = \frac{1}{s}, \qquad \mathscr{L}\{e^{4t}\}(s) = \frac{1}{s-4}, \qquad \mathscr{L}\{\sin(2t)\}(s) = \frac{2}{s^2+4}.$$

Therefore, we have

$$\begin{aligned} \mathscr{L}\{11 + 5e^{4t} - 6\sin(2t)\} &= 11\left(\frac{1}{s}\right) + 5\left(\frac{1}{s-4}\right) - 6\left(\frac{2}{s^2+4}\right) \\ &= \frac{11}{s} + \frac{5}{s-4} - \frac{12}{s^2+4}. \end{aligned}$$

The largest interval over which these functions are defined is $s > 4$. ◊

Existence of the Transform

Because the definition of the Laplace transform involves an improper integral, a major question to consider is whether the integral will converge for at least some values of s. Indeed, there are examples of functions for which the integral (1) fails to converge for any s-value, and so these do not have a Laplace transform: consider $f(t) = 1/t, g(t) = e^{t^2}$. What behavior of a function causes the integral to diverge? The function f grows too quickly near zero, where the function "blows up"; similarly, the function g grows too quickly as $t \to \infty$. So it seems that there will be some condition that bounds the growth of a function in order for its Laplace transform to exist.

Before stating the formal result, we recall some definitions from calculus. A function $f(t)$ is said to have a **jump discontinuity** at t_0 if f is discontinuous at t_0 but the one-sided limits $\lim\limits_{t\to t_0^-} f(t)$, $\lim\limits_{t\to t_0^+} f(t)$ exist (are finite). This is related to the concept of piecewise continuity.

Definition. A function $f(t)$ is piecewise continuous on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$ except possibly for a finite number of points at which f has a jump discontinuity. We say f is piecewise continuous on $[0, \infty)$ if f is piecewise continuous on $[0, N]$ for all $N > 0$.

Example 6. Show that

$$f(t) = \begin{cases} t, & 0 < t < 1, \\ 2, & 1 < t < 2, \\ (t-2)^2, & 2 \le t \le 3. \end{cases}$$

is piecewise continuous on $[0, 3]$.

Solution. The graph shows that f is continuous on every point of the interval except $t = 0, 1, 2$. Since all the one-sided limits are finite, each of these is a jump discontinuity. Therefore, f is piecewise continuous on $[0, 3]$. $\Diamond$

We now state the conditions necessary for the Laplace transform to exist.

Theorem 2. *If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathscr{L}\{f\}(s)$ exists for $s > \alpha$.*

The phrase "of exponential order α" in Theorem 2 basically means that the function grows no faster than the exponential function $e^{\alpha t}$. The functions frequently encountered in solving linear DE's with constant coefficients (namely, polynomials, exponentials, sines, and cosines) satisfy both these conditions, so that their Laplace transforms exist for sufficiently large values of s. Moreover, most of the time we shall not compute the transforms by hand using (1), but rather look them up in a Laplace table.

Homework: p. 360 #1, 5-27 odd.