Section 7.3: Properties of the Laplace Transform

Simplifying Computation of Laplace Transforms

We saw in the last section that calculating a Laplace transform from the definition requires evaluation of an improper integral, which is not always a simple matter. In this section, we derive several properties of the Laplace transform that will allow us to compute some Laplace transforms much more quickly.

The first property is that multiplying a function by an exponential simply shifts the Laplace transform.

Theorem 1. *If the Laplace transform $\mathscr{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then*

$$\mathscr{L}\{e^{at}f(t)\}(s) = F(s-a) \tag{1}$$

for $s > \alpha + a$.

Proof. We have

$$\mathscr{L}\{e^{at}f(t)\}(s) = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a). \qquad \square$$

Example 1. Determine the Laplace transform of $e^{at}\sin(bt)$.

Solution. In Section 7.2, we calculated $\mathscr{L}\{\sin(bt)\}(s) = F(s) = \dfrac{b}{s^2+b^2}$. Thus, by Theorem 1 we have

$$\mathscr{L}\{e^{at}\sin(bt)\}(s) = F(s-a) = \frac{b}{(s-a)^2+b^2}. \qquad \Diamond$$

A second property allows us to easily find the Laplace transform of the derivative of a function from the function's Laplace transform.

Theorem 2. *Let $f(t)$ be continuous on $[0,\infty)$ and $f'(t)$ be piecewise continuous on $[0,\infty)$, both of exponential order α. Then for $s > \alpha$,*

$$\mathscr{L}\{f'\}(s) = s\,\mathscr{L}\{f\}(s) - f(0). \tag{2}$$

Proof. Using integration by parts with $u = e^{-st}$ and $dv = f'(t)dt$, we get

$$\begin{aligned}
\mathscr{L}\{f'\}(s) &= \int_0^\infty e^{-st}f'(t)dt = \lim_{N\to\infty}\int_0^N e^{-st}f'(t)dt \\
&= \lim_{N\to\infty}\left[e^{-st}f(t)\Big|_0^N + s\int_0^N e^{-st}f(t)dt\right] \\
&= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\lim_{N\to\infty}\int_0^N e^{-st}f(t)dt \\
&= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\,\mathscr{L}\{f\}(s).
\end{aligned}$$

The fact that f is of exponential order α means $\lim_{N\to\infty} e^{-sN} f(N) = 0$, so the above equation reduces to

$$\mathscr{L}\{f'\}(s) = s\,\mathscr{L}\{f\}(s) - f(0). \qquad \square$$

We can extend this theorem by continuing to apply it to higher-order derivatives. For example,

$$\mathscr{L}\{f''\}(s) = s\,\mathscr{L}\{f'\}(s) - f'(0) = s[s\,\mathscr{L}\{f\}(s) - f(0)] - f'(0) = s^2\,\mathscr{L}\{f\}(s) - sf(0) - f'(0).$$

The general result is the following.

Theorem 3. *Let $f(t), f'(t), ..., f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$ with all functions of exponential order α. Then for $s > \alpha$,*

$$\mathscr{L}\{f^{(n)}\}(s) = s^n\,\mathscr{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0). \tag{3}$$

Loosely speaking, these theorems tell us that the Laplace transform allows us to replace differentiation with multiplication, which is why it is useful for solving IVP's.

Example 2. Use the formula $\mathscr{L}\{\sin(bt)\}(s) = \dfrac{b}{s^2 + b^2}$ to determine $\mathscr{L}\{\cos(bt)\}$.

Solution. Let $f(t) = \sin(bt)$, so that $f(0) = 0$ and $f'(t) = b\cos(bt)$. Then by Theorem 2, we have

$$\begin{aligned}
\mathscr{L}\{f'\}(s) &= s\,\mathscr{L}\{f\}(s) - f(0) \\
\mathscr{L}\{b\cos(bt)\}(s) &= s\,\mathscr{L}\{\sin(bt)\}(s) - 0 \\
b\,\mathscr{L}\{\cos(bt)\}(s) &= \frac{sb}{s^2 + b^2} \\
\mathscr{L}\{\cos(bt)\}(s) &= \frac{s}{s^2 + b^2}. \quad \diamondsuit
\end{aligned}$$

It is clear that the Laplace transform of the derivative is not the derivative of the Laplace transform. But it turns out that the derivative of the Laplace transform of a function is a Laplace transform of a related function. The following theorem makes this precise.

Theorem 4. *Let $F(s) = \mathscr{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Then for $s > \alpha$,*

$$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s). \tag{4}$$

Proof. We shall prove the statement for the case $n = 1$; the general result then follows from a proof method called induction. We take the derivative of the Laplace transform, and because of the assumptions on $f(t)$, we may switch the order of integration and differentiation:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)dt \Rightarrow \\
\frac{dF}{ds}(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt \\
&= \int_0^\infty \frac{d}{ds}(e^{-st}) f(t)dt \\
&= -\int_0^\infty e^{-st} t f(t)dt \\
&= -\mathscr{L}\{tf(t)\}(s) \Rightarrow \\
\mathscr{L}\{tf(t)\}(s) &= (-1)\frac{dF}{ds}(s). \quad \square
\end{aligned}$$

Example 3. Determine $\mathscr{L}\{t\sin(bt)\}$.

Solution. Recall that $\mathscr{L}\{\sin(bt)\}(s) = F(s) = \dfrac{b}{s^2+b^2}$. Then

$$F'(s) = \frac{-2bs}{(s^2+b^2)^2}$$

and so by Theorem 4, we have

$$\mathscr{L}\{t\sin(bt)\}(s) = -F'(s) = \frac{2bs}{(s^2+b^2)^2}. \qquad \Diamond$$

Homework: p. 365 #1-9 odd, 13-21 odd, 24, 25.