Section 7.4: Inverse Laplace Transform

A natural question to ask about any function is whether it has an inverse function. We now ask this question about the Laplace transform: given a function $F(s)$, will there be a function $f(t)$ such that $F(s) = \mathscr{L}\{f\}(s)$? It turns out that there is at most one continuous function $f(t)$ which satisfies this property (there could be infinitely many discontinuous functions with the same Laplace transform, but we prefer to work with continuous functions).

Definition. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0,\infty)$ and satisfies $\mathscr{L}\{f\} = F$, then we say that $f(t)$ is the inverse Laplace transform of $F(s)$ and employ the notation $f = \mathscr{L}^{-1}\{F\}$.

This idea has more than theoretical interest, however; we'll see in the next section that finding inverse Laplace transforms is a critical step in solving initial value problems. To determine the inverse Laplace transform of a function, we try to match it with the form of an entry in the right-hand column of a Laplace table.

Example 1. Determine $\mathscr{L}^{-1}\{F\}$ for
(a) $F(s) = \dfrac{2}{s^3}$, (b) $F(s) = \dfrac{3}{s^2+9}$, (c) $F(s) = \dfrac{s-1}{s^2-2s+5}$.

Solution. (a) $\mathscr{L}^{-1}\left\{\dfrac{2}{s^3}\right\}(t) = \mathscr{L}^{-1}\left\{\dfrac{2!}{s^3}\right\}(t) = t^2$

(b) $\mathscr{L}^{-1}\left\{\dfrac{3}{s^2+9}\right\}(t) = \mathscr{L}^{-1}\left\{\dfrac{3}{s^2+3^2}\right\}(t) = \sin(3t)$

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-1}{s^2-2s+5}\right\}(t) = \mathscr{L}^{-1}\left\{\dfrac{s-1}{(s-1)^2+2^2}\right\}(t) = e^t\cos(2t)$. ◊

Of course, very often the transform we are given will not correspond exactly to an entry in the Laplace table. One tool we can use in handling more complicated functions is the linearity of the inverse Laplace transform, a property it inherits from the original Laplace transform.

Theorem 1. *Assume that $\mathscr{L}^{-1}\{F\}, \mathscr{L}^{-1}\{F_1\}$, and $\mathscr{L}^{-1}\{F_2\}$ exist and are continuous on $[0,\infty)$ and let c be any constant. Then*

$$\mathscr{L}^{-1}\{F_1+F_2\} = \mathscr{L}^{-1}\{F_1\} + \mathscr{L}^{-1}\{F_2\},$$

$$\mathscr{L}^{-1}\{cF\} = c\,\mathscr{L}^{-1}\{F\}.$$

Example 2. Determine $\mathscr{L}^{-1}\left\{\dfrac{5}{s-6} - \dfrac{6s}{s^2+9} + \dfrac{3}{2s^2+8s+10}\right\}$.

Solution. By linearity, we have

$$\mathscr{L}^{-1}\left\{\frac{5}{s-6} - \frac{6s}{s^2+9} + \frac{3}{2s^2+8s+10}\right\} = 5\,\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\} - 6\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{3}{2}\,\mathscr{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}.$$

Rewriting the final term using completing the square, it becomes $\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+1^2}\right\}$. Therefore, by the Laplace table we see that

$$\mathscr{L}^{-1}\left\{\frac{5}{s-6}-\frac{6s}{s^2+9}+\frac{3}{2s^2+8s+10}\right\}(t)=5e^{6t}-6\cos(3t)+\frac{3}{2}e^{-2t}\sin t. \qquad \Diamond$$

Example 3. Determine $\mathscr{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}$.

Solution. The fourth power in the denominator suggests that the inverse Laplace transform is of the form

$$\mathscr{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\}(t)=e^{at}t^n.$$

In this case, $a=-2, n=3$, so by linearity we have

$$\mathscr{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}(t)=\frac{5}{6}\mathscr{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\}(t)=\frac{5}{6}e^{-2t}t^3. \qquad \Diamond$$

Example 4. Determine $\mathscr{L}^{-1}\left\{\frac{3s+2}{s^2+2s+10}\right\}$.

Solution. Using completing the square, the denominator can be rewritten as

$$s^2+2s+10=s^2+2s+1+9=(s+1)^2+3^2.$$

Therefore, the form of $F(s)$ suggests the following two formulas from the Laplace table:

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}(t) &= e^{at}\cos(bt),\\ \mathscr{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\}(t) &= e^{at}\sin(bt),\end{aligned}$$

where we have $a=-1, b=3$. Thus, we want to write

$$\frac{3s+2}{s^2+2s+10}=A\frac{s+1}{(s+1)^2+3^2}+B\frac{3}{(s+1)^2+3^2}$$

for an appropriate choice of constants A, B. By clearing the denominator, we have the equation

$$3s+2=A(s+1)+3B=As+(A+3B).$$

Equating coefficients gives us $A=3, B=-1/3$, so by linearity, we have

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{3s+2}{s^2+2s+10}\right\}(t) &= 3\,\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\}(t)-\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{(s+1)^2+3^2}\right\}(t)\\ &= 3e^{-t}\cos(3t)-\frac{1}{3}e^{-t}\sin(3t). \quad \Diamond\end{aligned}$$

In Example 4, we found it easier to take the Laplace transform of the broken up fraction than of the original combined fraction. The procedure we used above, which you may recall from Calculus 2 and which will prove very useful in simplifying these functions, is called the **method of partial fractions**. Because of its importance, we now review this method.

Review of Partial Fractions

A rational function $P(s)/Q(s)$, where $P(s), Q(s)$ are polynomials with the degree of P less than the degree of Q, has a partial fraction expansion based on the factorization of Q. There are three main cases to consider, the easiest being that of nonrepeated linear factors. If $Q(s)$ is a product of distinct linear factors,

$$Q(s) = (s - r_1)(s - r_2)\cdots(s - r_n),$$

then the partial fraction expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \frac{A_2}{s - r_2} + \cdots + \frac{A_n}{s - r_n}.$$

The constants $A_1, ..., A_n$ can be derived by clearing denominators and either equating coefficients of the resulting polynomials as in Example 4, or by careful substitution. We demonstrate the second way in the following examples.

Example 5. Determine $\mathscr{L}^{-1}\{F\}$ for $F(s) = \dfrac{7s - 1}{(s+1)(s+2)(s-3)}$.

Solution. Since the denominator has three distinct linear factors, the partial fraction expansion has the form

$$\frac{7s - 1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3},$$

and upon clearing denominators (multiplying through by $(s+1)(s+2)(s-3)$), we have

$$(1) \qquad 7s - 1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2).$$

If we substitute s-values which make the linear factors zero, then each time two terms will drop out and it will be simple to solve for each coefficient. First let $s = -1$; then (1) becomes

$$-8 = -4A \Rightarrow A = 2.$$

Similarly, letting $s = -2$ gives

$$-15 = 5B \Rightarrow B = -3,$$

and letting $s = 3$ yields

$$20 = 20C \Rightarrow C = 1.$$

Therefore, we can now use linearity to calculate the inverse Laplace transform:

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s+1}-\frac{3}{s+2}+\frac{1}{s-3}\right\}(t)\\ &= 2\,\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)-3\,\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}(t)+\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\}(t)\\ &= 2e^{-t}-3e^{-2t}+e^{3t}. \quad \diamondsuit\end{aligned}$$

The second scenario involves repeated linear factors. If the highest power of $s-r$ that divides $Q(s)$ is $(s-r)^m$, then the portion of the partial fraction expansion corresponding to $(s-r)^m$ is

$$\frac{A_1}{s-r}+\frac{A_2}{(s-r)^2}+\cdots+\frac{A_m}{(s-r)^m}.$$

Example 6. Determine $\mathscr{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

Solution. The partial fraction expansion has the form

$$\frac{s^2+9s+2}{(s-1)^2(s+3)}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+3}.$$

Clearing denominators gives

$$s^2+9s+2=A(s-1)(s+3)+B(s+3)+C(s-1)^2. \tag{2}$$

Again, we substitute values which make the linear factors vanish. Letting $s=1$ in (2) gives

$$12=4B\Rightarrow B=3,$$

and letting $s=-3$ yields

$$-16=16C\Rightarrow C=-1.$$

To find the third constant, we can plug in any other value for s; a convenient choice is $s=0$:

$$\begin{aligned}2 &= -3A+3B+C=-3A+9-1\\ -6 &= -3A\Rightarrow A=2.\end{aligned}$$

Therefore, the inverse Laplace transform is

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s-1}+\frac{3}{(s-1)^2}-\frac{1}{s+3}\right\}(t)\\ &= 2\,\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}(t)+3\,\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t)-\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}(t)\\ &= 2e^{t}+3te^{t}-e^{-3t}. \quad \diamondsuit\end{aligned}$$

The most difficult case is that of a quadratic factor. If m is the highest power of $(s-\alpha)^2+\beta^2$ that divides $Q(s)$, then the partial fraction expansion for that term is

$$\frac{A_1s+B_1}{(s-\alpha)^2+\beta^2}+\frac{A_2s+B_2}{[(s-\alpha)^2+\beta^2]^2}+\cdots+\frac{A_ms+B_m}{[(s-\alpha)^2+\beta^2]^m}.$$

However, for looking up Laplace transforms it is more convenient to have numerators of the form $A_i(s-\alpha)+\beta B_i$, so we can equivalently write

$$\frac{A_1(s-\alpha)+\beta B_1}{(s-\alpha)^2+\beta^2}+\frac{A_2(s-\alpha)+\beta B_2}{[(s-\alpha)^2+\beta^2]}+\cdots+\frac{A_m(s-\alpha)+\beta B_m}{[(s-\alpha)^2+\beta^2]^m}.$$

Example 7. Determine $\mathscr{L}^{-1}\left\{\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

Solution. Since the quadratic factor in the denominator is irreducible, we rewrite using completing the square: $s^2-2s+5=(s-1)^2+2^2$. Then the partial fraction expansion has the form

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)}=\frac{A(s-1)+2B}{(s-1)^2+2^2}+\frac{C}{s+1}.$$

Clearing denominators gives

$$2s^2+10s=[A(s-1)+2B](s+1)+C(s^2-2s+5). \tag{3}$$

There are two linear terms involved in this expression, so pick the two s-values which make them vanish. When $s=-1$, we get

$$-8=8C\Rightarrow C=-1.$$

When $s=1$, we get

$$12=4B+4C=4B-4\Rightarrow 4B=16\Rightarrow B=4.$$

For the final constant, set $s=0$:

$$0=-A+2B+5C\Rightarrow A=2(4)+5(-1)=3.$$

Therefore, we find the inverse Laplace transform to be

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{3(s-1)+2(4)}{(s-1)^2+2^2}-\frac{1}{s+1}\right\}\\ &= 3\,\mathscr{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\}+4\,\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}-\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}\\ &= 3e^t\cos(2t)+4e^t\sin(2t)-e^{-t}. \quad \Diamond\end{aligned}$$

Homework: pp. 374-375, #1-29 odd.