## Section 7.5: Solving Initial Value Problems

We have spent a few sections looking exclusively at the Laplace transform with no mention of differential equations. We now show how the Laplace transform can be used to solve initial value problems for linear DE's. One might wonder whether the new technique is worth all the time spent developing it. The advantage here is that unlike the methods in Chapter 4, we can solve IVP's *without finding a general solution first*. In addition, this technique can handle some equations which the previous methods could not: nonhomogeneous equations with forcing functions which have jump discontinuities, some linear DE's with variable coefficients, systems of DE's, and even partial differential equations.

Algorithm. To solve an IVP using Laplace transforms:

- (1) Take the Laplace transform of both sides.
- (2) Using properties of the Laplace transform and the initial conditions, solve the resulting equation for the transform.
- (3) Find the inverse Laplace transform of both sides.

We now present a few examples to familiarize ourselves with this new technique.

**Example 1.** Solve the IVP  $y'' - 2y' + 5y = -8e^{-t}$ , y(0) = 2, y'(0) = 12.

Solution. Taking the Laplace transform of both sides and using linearity, we have

$$\mathscr{L}\{y'' - 2y' + 5y\} = \mathscr{L}\{-8e^{-t}\}$$
(1) 
$$\mathscr{L}\{y''\}(s) - 2\mathscr{L}\{y'\}(s) + 5\mathscr{L}\{y\}(s) = \frac{-8}{s+1}.$$

For convenience, set  $Y(s) := \mathscr{L}{y}(s)$ . We calculate the Laplace of the derivatives using the formulas from Section 7.3:

$$\begin{aligned} \mathscr{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) - 2, \\ \mathscr{L}\{y''\}(s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 12. \end{aligned}$$

Substituting these values into (1), we then solve for Y(s):

$$[s^{2}Y(s) - 2s - 12] - 2[sY(s) - 2] + 5Y(s) = \frac{-8}{s+1}$$

$$(s^{2} - 2s + 5)Y(s) = 2s + 8 - \frac{8}{s+1}$$

$$(s^{2} - 2s + 5)Y(s) = \frac{2s^{2} + 10s}{s+1}$$

$$Y(s) = \frac{2s^{2} + 10s}{(s^{2} - 2s + 5)(s+1)}.$$

Next we need to take the inverse Laplace transform, which requires partial fractions; fortunately, we already did this in Example 7 of the previous section, where we got

$$y(t) = 3e^t \cos(2t) + 4e^t \sin(2t) - e^{-t}.$$

You can easily check your work by making sure the function you obtain satisfies the given initial conditions.

**Example 2.** Solve the IVP  $y'' + 4y' - 5y = te^t$ , y(0) = 1, y'(0) = 0.

**Solution.** Again, we denote  $\mathscr{L}{y}(s)$  by Y(s). Taking the Laplace transform of both sides gives

(2) 
$$\mathscr{L}\{y''\}(s) + 4\mathscr{L}\{y'\}(s) - 5Y(s) = \mathscr{L}\{te^t\} = \frac{1}{(s-1)^2}.$$

As before, we use the initial conditions to rewrite the Laplace of the derivatives:

$$\mathscr{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 1, \mathscr{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - sy(s) - sy(s)$$

Substituting into (2) and solving for Y(s), we get

$$[s^{2}Y(s) - s] + 4[sY(s) - 1] - 5Y(s) = \frac{1}{(s - 1)^{2}}$$
$$(s^{2} + 4s - 5)Y(s) = s + 4 + \frac{1}{(s - 1)^{2}}$$
$$(s + 5)(s - 1)Y(s) = \frac{s^{3} + 2s^{2} - 7s + 5}{(s - 1)^{2}}$$
$$Y(s) = \frac{s^{3} + 2s^{2} - 7s + 5}{(s + 5)(s - 1)^{3}}.$$

We now use partial fractions to write Y(s) in the form

(3) 
$$\frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3} = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}$$

Clearing denominators gives

$$s^{3} + 2s^{2} - 7s + 5 = A(s-1)^{3} + B(s+5)(s-1)^{2} + C(s+5)(s-1) + D(s+5).$$

Setting s = 1 gives D = 1/6, and setting s = -5 gives A = 35/216. To get B and C, set s = 0, s = -1 to get the system of equations

$$\frac{-35}{216} + 5B - 5C + \frac{5}{6} = 5 \Rightarrow 5B - 5C = \frac{935}{216}$$
$$\frac{-280}{216} + 16B - 8C + \frac{2}{3} = 13 \Rightarrow 16B - 8C = \frac{368}{27}.$$

Solving this system gives B = 181/216, C = -1/36, so substituting into (3) we have

$$Y(s) = \frac{35}{216} \left(\frac{1}{s+5}\right) + \frac{181}{216} \left(\frac{1}{s-1}\right) - \frac{1}{36} \left(\frac{1}{(s-1)^2}\right) + \frac{1}{12} \left(\frac{2}{(s-1)^3}\right).$$

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It only remains to take the inverse Laplace transform:

$$y(t) = \frac{35}{216}e^{-5t} + \frac{181}{216}e^t - \frac{1}{36}te^t + \frac{1}{12}t^2e^t.$$

**Example 3.** Solve the IVP  $w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}$ ,  $w(\pi) = 2$ ,  $w'(\pi) = 12$ .

**Solution.** We can only simplify the Laplace of derivatives if the initial conditions are given at 0, so we shift the problem by setting  $y(t) := w(t + \pi)$ . Then  $y'(t) = w'(t + \pi)$  and  $y''(t) = w''(t + \pi)$ , so replacing t with  $t + \pi$  in the problem gives

$$w''(t+\pi) - 2w'(t+\pi) + 5w(t+\pi) = -8e^{\pi - (t+\pi)} = -8e^{-t}$$

and thus

$$y''(t) - 2y'(t) + 5y(t) = -8e^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

Observe that in Example 1, we solved this DE and got the solution

(4) 
$$y(t) = 3e^t \cos(2t) + 4e^t \sin(2t) - e^{-t}$$

Since  $w(t) = y(t - \pi)$ , we replace t with  $t - \pi$  in (4) to get

$$w(t) = 3e^{t-\pi} \cos[2(t-\pi)] + 4e^{t-\pi} \sin[2(t-\pi)] - e^{-(t-\pi)}$$
  
=  $3e^{t-\pi} \cos(2t) + 4e^{t-\pi} \sin(2t) - e^{\pi-t}.$ 

So far we have only applied the Laplace method to linear DE's with constant coefficients. But we can also effectively solve equations whose coefficients are polynomials in t by using another property we derived; recall that

(5) 
$$\mathscr{L}\lbrace t^n f(t)\rbrace(s) = (-1)^n \frac{d^n F}{ds^n}(s).$$

For f(t) = y'(t) and n = 1 in (5), we get

(6) 
$$\mathscr{L}\{ty'(t)\}(s) = -\frac{d}{ds}\mathscr{L}\{y'\}(s)$$
$$= -\frac{d}{ds}[sY(s) - y(0)] = -sY'(s) - Y(s).$$

If we put f(t) = y''(t) instead, we get

$$\begin{aligned} \mathscr{L}\{ty''(t)\}(s) &= -\frac{d}{ds} \mathscr{L}\{y''\}(s) \\ &= -\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] \\ &= -s^2Y'(s) - 2sY(s) + y(0). \end{aligned}$$

What does this mean? If a linear DE in y(t) has coefficients which are polynomials of degree 1 in t, then whatever the order of the original equation, the DE for Y(s) after taking Laplace transforms is a linear *first-order* equation, which we know how to solve. To determine the resulting constant of integration, we use one further property of the Laplace transform: if f(t) is piecewise continuous on  $[0, \infty)$  and of exponential order, then  $\lim_{s\to\infty} \mathscr{L}{f}(s) = 0$ . 4

**Example 4.** Solve the IVP y'' + 2ty' - 4y = 1, y(0) = y'(0) = 0.

**Solution.** As usual, we put  $Y(s) = \mathscr{L}{y}(s)$  and take the Laplace transform of both sides:

(7) 
$$\mathscr{L}\{y''\}(s) + 2\mathscr{L}\{ty'(t)\}(s) - 4Y(s) = \frac{1}{s}$$

Using the initial conditions and formula (6), we have

$$\mathscr{L}\{y''\}(s)=s^2Y(s)-sy(0)-y'(0)=s^2Y(s), \mathscr{L}\{ty'(t)\}(s)=-sY'(s)-Y(s).$$
 Substituting into (7) yields

$$s^{2}Y(s) + 2[-sY'(s) - Y(s)] - 4Y(s) = \frac{1}{s}$$
$$-2sY'(s) + (s^{2} - 6)Y(s) = \frac{1}{s}$$
$$Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right)Y(s) = \frac{-1}{2s^{2}}$$

We can solve this linear equation by finding the integrating factor

$$\mu(s) = e^{\int (3/s - s/2)ds} = e^{3\ln s - s^2/4} = s^3 e^{-s^2/4}.$$

Then

$$\begin{aligned} \frac{d}{ds} \{\mu(s)Y(s)\} &= \frac{d}{ds} \{s^3 e^{-s^2/4} Y(s)\} = -\frac{s}{2} e^{-s^2/4} \\ s^3 e^{-s^2/4} Y(s) &= e^{-s^2/4} + C \\ Y(s) &= \frac{1}{s^3} + C \frac{e^{s^2/4}}{s^3}. \end{aligned}$$

Now since the limit of Y(s) as  $s \to \infty$  should be 0 and the second term in Y(s) grows to infinity, we must take C = 0 and thus  $Y(s) = \frac{1}{s^3} = \frac{1}{2} \left(\frac{2}{s^3}\right)$ . Therefore, taking the inverse Laplace transform gives

$$y(t) = \frac{1}{2}t^2.$$

Homework: pp. 382-383 #1-21 odd, 25, 27, 33-35, 38.