Section 9.1: Introduction to Matrix Methods for Linear Systems

Representing Systems in Matrix Form

In this chapter we focus on systems of differential equations which are all linear; an advantage of this property is that such systems can be compactly represented in matrix form. This is not only convenient for notation, but also leads to new and efficient techniques for obtaining solutions to these systems.

In general, if we have a system of differential equations given by

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n, \end{aligned}$$

the matrix form of this system is

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is the **coefficient matrix**

$$\mathbf{A} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

and $\mathbf{x}$ is the solution vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Such a system is called a linear homogeneous system in **normal form**.

Example 1. Express the following system as a matrix equation:

$$\begin{aligned} x_1' &= 2x_1 + t^2x_2 + (4t + e^t)x_4, \\ x_2' &= (\sin t)x_2 + (\cos t)x_3, \\ x_3' &= x_1 + x_2 + x_3 + x_4, \\ x_4' &= 0. \end{aligned}$$

Solution. The matrix form is given by

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 2 & t^2 & 0 & 4t + e^t \\ 0 & \sin t & \cos t & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \qquad \lozenge$$

If we have an nth order linear homogeneous DE

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = 0,$$

this can be written as a linear system in normal form by defining the first $(n-1)$ derivatives of y (including y itself) as unknowns: $x_1(t) = y(t), x_2(t) = y'(t), \ldots, x_n(t) = y^{(n-1)}(t)$. Solving the DE for $y^{(n)}(t)$ yields the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\vdots \\ x_{n-1}' &= x_n, \\ x_n' &= -\frac{a_0(t)}{a_n(t)}x_1 - \frac{a_1(t)}{a_n(t)}x_2 - \cdots - \frac{a_{n-1}(t)}{a_n(t)}x_n. \end{aligned}$$

Example 2. Express the DE for the undamped, unforced mass-spring oscillator $my'' + ky = 0$ as an equivalent system of first-order equations in normal form, using matrix notation.

Solution. We write $y' = v$ so that $y'' = v'$, and the original equation becomes $mv' + ky = 0$. Solving this for v' yields $v' = -\frac{k}{m}y$, so the matrix normal form of the system is

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}. \qquad \Diamond$$

Systems of two or more higher-order DE's can be treated in the same way, applying this procedure to each unknown function as in the following example.

Example 3. A coupled mass-spring oscillator is governed by the system

$$\begin{aligned} 2\frac{d^2x}{dt^2} + 6x - 2y &= 0, \\ \frac{d^2y}{dt^2} + 2y - 2x &= 0. \end{aligned}$$

Write this in matrix notation.

Solution. We write the lower-order derivatives as the unknowns: $x_1 = x, x_2 = x', x_3 = y, x_4 = y'$. In these variables, the given system becomes

$$\begin{aligned} 2x_2' + 6x_1 - 2x_3 &= 0, \\ x_4' + 2x_3 - 2x_1 &= 0. \end{aligned}$$

Putting this in normal form, we have

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -3x_1 + x_3, \\ x_3' &= x_4, \\ x_4' &= 2x_1 - 2x_3. \end{aligned}$$

Therefore, the matrix notation is

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \qquad \Diamond$$

Homework: p. 502, #1-13 odd.