Section 9.4: Linear Systems in Normal Form

We will now introduce some terminology and notation for systems which parallel the definitions for individual DE's from earlier chapters. A linear system of n DE's is in **normal form** if it can be written as

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \tag{1}$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$, and $\mathbf{A}$ is an $n \times n$ matrix. If $\mathbf{f}(t) = \mathbf{0}$, the system is called **homogeneous**; otherwise, it is called **nonhomogeneous**. If the elements of the coefficient matrix $\mathbf{A}$ are all constants, the system has **constant coefficients**. Recall from Section 9.1 that we can rewrite an n-th order linear DE

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_0(t)y(t) = g(t)$$

in the form of equation (1) where $\mathbf{f}(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}$ and

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & \cdots & -p_{n-2}(t) & -p_{n-1}(t) \end{bmatrix}.$$

As one might hope, the theory for systems in normal form is very similar to the theory of linear DE's in Chapters 4 and 6. As before, the notion of linear independence is important for writing general solutions.

Definition. The vector functions $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are said to be linearly dependent on an interval I if there exist constants $c_1, \ldots, c_m$ not all zero such that

$$c_1\mathbf{x}_1(t) + \cdots + c_m\mathbf{x}_m(t) = \mathbf{0}$$

for all $t \in I$. Otherwise, they are said to be linearly independent on I.

Example 1. Show that the functions $\mathbf{x}_1(t) = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$, $\mathbf{x}_2(t) = \begin{bmatrix} 3e^t \\ 0 \\ 3e^t \end{bmatrix}$, and $\mathbf{x}_3(t) = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$ are linearly dependent on $(-\infty, \infty)$.

Solution. Since $\mathbf{x}_2(t)$ is a scalar multiple of $\mathbf{x}_1(t)$, we see that

$$3\mathbf{x}_1(t) - \mathbf{x}_2(t) + 0 \cdot \mathbf{x}_3(t) = \mathbf{0}$$

for all t. Therefore, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly dependent. $\diamondsuit$

To prove linear independence, the technique is to assume some linear combination equals $\mathbf{0}$, and deduce that every coefficient must be zero, as illustrated in the following example.

Example 2. Show that the functions $\mathbf{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$, $\mathbf{x}_2(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ -e^{2t} \end{bmatrix}$, and $\mathbf{x}_3(t) = \begin{bmatrix} e^{t} \\ 2e^{t} \\ e^{t} \end{bmatrix}$ are linearly independent on $(-\infty, \infty)$.

Solution. Assume there are constants c_1, c_2, c_3 such that $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t) = \mathbf{0}$ for every t. Substituting $t = 0$, we obtain the system of linear equations

$$\begin{aligned} c_1 + c_2 + c_3 &= 0, \\ c_2 + 2c_3 &= 0, \\ c_1 - c_2 + c_3 &= 0, \end{aligned}$$

which has solution $c_1 = c_2 = c_3 = 0$. Thus, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent. $\diamondsuit$

We can define a determinant which is related to the scalar case, which can be useful for determining linear independence.

Definition. The <u>Wronskian</u> of n vector functions $\mathbf{x}_1(t) = \operatorname{col}(x_{1,1}, \ldots, x_{n,1}), \ldots, \mathbf{x}_n(t) = \operatorname{col}(x_{1,n}, \ldots, x_{n,n})$ is the real-valued function

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \begin{vmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{vmatrix}.$$

Similarly to previous discussions, a set of vector solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ to a homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are linearly independent on I if and only if their Wronskian is nonzero for at least one point in I. The general solution for a homogeneous system is described in the following theorem.

Theorem 1. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be linearly independent solutions to the homogeneous system*

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{2}$$

on the interval I, where $\mathbf{A}(t)$ is an $n \times n$ matrix function continuous on I. Then a general solution to (2) on I is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t),$$

for constants $c_1, \ldots, c_n$.

The set of linearly independent solutions $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is called a **fundamental solution set** for (2); if we let each of these vectors form the column of a matrix, we obtain the **fundamental matrix**

$$\mathbf{X}(t) = \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{bmatrix}.$$

Example 3. Verify that the set

$$S = \left\{ \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}, \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix} \right\}$$

is a fundamental solution set for the system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) \tag{3}$$

on the interval $(-\infty, \infty)$ and find a fundamental matrix and a general solution for (3).

Solution. Substituting the vectors from S into (3), we obtain

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \mathbf{x}'(t),$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix} = \mathbf{x}'(t),$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} 0 \\ -e^{-t} \\ e^{-t} \end{bmatrix} = \mathbf{x}'(t).$$

Thus, each vector is a solution. S will be a fundamental solution set if the vectors are linearly independent, or equivalently, if their Wronskian is never zero:

$$W(t) = \begin{vmatrix} e^{2t} & -e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & -e^{-t} \end{vmatrix} = e^{2t} \begin{vmatrix} 0 & e^{-t} \\ e^{-t} & -e^{-t} \end{vmatrix} + e^{-t} \begin{vmatrix} e^{2t} & e^{-t} \\ e^{2t} & -e^{-t} \end{vmatrix} = -3.$$

The fundamental matrix is the one we used to compute the Wronskian; that is,

$$\mathbf{X}(t) = \begin{bmatrix} e^{2t} & -e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & -e^{-t} \end{bmatrix}.$$

Finally, a general solution to (3) is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}. \qquad \Diamond$$

Remark. Instead of showing that each column $\mathbf{x}$ in S satisfies $\mathbf{x}' = \mathbf{A}\mathbf{x}$, it is equivalent to show that the fundamental matrix $\mathbf{X}$ satisfies $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

We also maintain the superposition principle from linear DE's. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to the nonhomogeneous systems with nonhomogeneities $\mathbf{g}_1$ and $\mathbf{g}_2$, respectively, then $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is a solution to the system with nonhomogeneity $c_1\mathbf{g}_1 + c_2\mathbf{g}_2$. This allows us to write general solutions for nonhomogeneous systems.

Theorem 2. *Let $\mathbf{x}_p$ be a particular solution to the nonhomogeneous system*

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{4}$$

on the interval I, and let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a fundamental solution set on I for the corresponding homogeneous system $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$. Then a general solution to (4) on I is

$$\mathbf{x}(t) = \mathbf{x}_p(t) + c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$

where $c_1, \ldots, c_n$ are constants.

In later sections, we shall explore how to find fundamental solution sets for homogeneous systems and particular solutions for nonhomogeneous systems.

Homework: pp. 523-524 #1-25 odd.