Section 9.5: Homogeneous Linear Systems with Constant Coefficients

Eigenvalues and Eigenvectors

We now show how to obtain a general solution for the homogeneous system
\[
\mathbf{x}'(t) = A\mathbf{x}(t)
\]
where \( A \) is a constant \( n \times n \) matrix. Recall from the previous section that this amounts to finding \( n \) linearly independent solutions to (1). In Chapter 4, we used the fact that homogeneous linear equations with constant coefficients had solutions of the form \( e^{rt} \); extending this idea, we guess that the system (1) will have solutions of the form \( \mathbf{x}(t) = \mathbf{u}e^{rt} \) for some constant \( r \) and constant vector \( \mathbf{u} \). Indeed, substituting this vector into (1) gives
\[
re^{rt}\mathbf{u} = A e^{rt}\mathbf{u} = e^{rt}A\mathbf{u}.
\]
Rearranging terms after canceling the exponential yields
\[
(A - rI)\mathbf{u} = 0,
\]
where \( I \) denotes the identity matrix.

Therefore, \( \mathbf{x}(t) = e^{rt}\mathbf{u} \) is a solution to (1) if and only if \( r \) and \( \mathbf{u} \) satisfy equation (2). This equation is trivially satisfied when \( \mathbf{u} = 0 \), but this is not part of any linearly independent set (why?), so we require also that \( \mathbf{u} \neq 0 \). There is a special name for such \( r \) and \( \mathbf{u} \).

**Definition.** Let \( A \) be an \( n \times n \) constant matrix. The **eigenvalues** of \( A \) are those (real or complex) numbers \( r \) for which \( (A - rI)\mathbf{u} = 0 \) has at least one nontrivial solution \( \mathbf{u} \). The corresponding nontrivial solutions \( \mathbf{u} \) are called the **eigenvectors** of \( A \) associated with \( r \).

A basic fact of linear algebra (mentioned in Section 9.3) is that (2) will have a nontrivial solution if and only if the determinant \( |A - rI| = 0 \). Since the determinant of this matrix is a polynomial in \( r \) of degree \( n \), call it \( p(r) \), to find the eigenvalues of a matrix \( A \) we must find the zeros of the **characteristic polynomial** \( p(r) \). This is similar to the auxiliary equation for scalar DE’s.

**Example 1.** Find the eigenvalues and eigenvectors of the matrix \( A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \).

**Solution.** We find the characteristic polynomial for \( A \):
\[
|A - rI| = \begin{vmatrix} 2 - r & -3 \\ 1 & -2 - r \end{vmatrix} = (2 - r)(-2 - r) + 3 = r^2 - 1 = 0.
\]
Therefore, the eigenvalues of \( A \) are \( r_1 = 1, r_2 = -1 \). For the eigenvectors corresponding to \( r_1 = 1 \), we solve the equation \( (A - I)\mathbf{u} = 0 \):
\[
\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
This is the equation \( u_1 = 3u_2 \), so if we set \( u_2 = s \), then \( \mathbf{u}_1 = s \begin{bmatrix} 3 \\ 1 \end{bmatrix} \).
For $r_2 = -1$, we solve $(A + I)u = 0$:

$$
\begin{bmatrix}
3 & -3 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

This is the equation $u_1 = u_2$, so if we set $u_2 = s$, then $u_2 = s \begin{bmatrix}1 \\ 1 \end{bmatrix}$.

Remark. The set of eigenvectors for $r_1$ forms a subspace of $\mathbb{R}^2$ when the zero vector is adjoined (and likewise for the set of eigenvectors for $r_2$). These subspaces are called eigenspaces.

Example 2. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix}1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$.

Solution. We find the characteristic polynomial:

$$
|A - rI| = \begin{vmatrix}
1 - r & 2 & -1 \\
1 & -r & 1 \\
4 & -4 & 5 - r
\end{vmatrix} = (r - 1)(r - 2)(r - 3) = 0.
$$

Therefore, the eigenvalues of $A$ are $r_1 = 1$, $r_2 = 2$, $r_3 = 3$. To find eigenvectors for $r_1 = 1$, solve $(A - I)u = 0$:

$$
\begin{bmatrix}
0 & 2 & -1 \\
1 & -1 & 1 \\
4 & -4 & 4
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
$$

Since the third row is a multiple of the second row, this is equivalent to the system

$$
u_1 - u_2 + u_3 = 0, $$
$$2u_2 - u_3 = 0.
$$

If we assign an arbitrary value $s$ to $u_2$, solving the second equation gives $u_3 = 2s$, and substituting these into the first equation gives $u_1 = -s$. Therefore, the eigenvectors for $r_1$ are $u_1 = s \begin{bmatrix}-1 \\ 1 \\ 2 \end{bmatrix}$.

For $r_2 = 2$, we solve

$$
\begin{bmatrix}
-1 & 2 & -1 \\
1 & -2 & 1 \\
4 & -4 & 3
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
$$

Now the first and second equations are multiples, so an equivalent system is

$$
u_1 - 2u_2 + u_3 = 0, $$
$$4u_1 - 4u_2 + 3u_3 = 0.
$$

Again letting $u_2 = s$ gives $u_3 = 4s$, $u_1 = -2s$, so the eigenvectors for $r_2$ are $u_2 = s \begin{bmatrix}-2 \\ 1 \\ 4 \end{bmatrix}$. 

Finally, for $r_3 = 3$ we solve
\[
\begin{bmatrix}
-2 & 2 & -1 \\
1 & -3 & 1 \\
4 & -4 & 2 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}.
\]
The first and third equations are multiples, so an equivalent system is
\[
u_1 - 3u_2 + u_3 = 0, \\
2u_1 - 2u_2 + u_3 = 0.
\]
Letting $u_2 = s$ gives $u_3 = 4s, u_1 = -s$, so the eigenvectors for $r_3$ are $u_3 = s \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$. ♦

The Case of $n$ Linearly Independent Eigenvectors

How does this calculation help us with the general solution to our DE? If our matrix has $n$ linearly independent eigenvectors, then we will have enough solutions to write the general solution, as stated in the following theorem.

**Theorem 1.** Suppose the $n \times n$ constant matrix $A$ has $n$ linearly independent eigenvectors $u_1, u_2, \ldots, u_n$. Let $r_i$ be the eigenvalue corresponding to $u_i$. Then $\{e^{r_1t}u_1, e^{r_2t}u_2, \ldots, e^{r_nt}u_n\}$ is a fundamental solution set on $(-\infty, \infty)$ for the homogeneous system $x' = Ax$, which has general solution
\[
x(t) = c_1e^{r_1t}u_1 + c_2e^{r_2t}u_2 + \cdots + c_ne^{r_nt}u_n,
\]
where $c_1, \ldots, c_n$ are arbitrary constants.

**Example 3.** Find a general solution of $x'(t) = Ax(t)$, where $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

**Solution.** In Example 1, we found that this matrix has eigenvalues $r_1 = 1, r_2 = -1$ with eigenvectors $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where we have taken $s = 1$. Since $u_1$ and $u_2$ are linearly independent, a general solution is
\[
x(t) = c_1e^t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

The next theorem is very useful for confirming linear independence of eigenvectors.

**Theorem 2.** If $r_1, \ldots, r_m$ are distinct eigenvalues for the matrix $A$ and $u_i$ is an eigenvector associated with $r_i$, then $u_1, \ldots, u_m$ are linearly independent.

**Proof.** We only prove the case $m = 2$; the more general result follows by induction. Suppose by way of contradiction that $u_1$ and $u_2$ are linearly dependent; that is, for some constant $c$ we have
\[
u_1 = cu_2.
\]
Multiplying (4) through by $A$, we have
\[ Au_1 = cAu_2 \Rightarrow r_1 u_1 = cr_2 u_2. \]

Multiplying (4) by $r_2$ and subtracting this from (5) gives
\[ (r_1 - r_2)u_1 = 0. \]

Since $u_1 \neq 0$, this implies $r_1 = r_2$, but this contradicts the assumption that the eigenvalues were distinct. \qed

This means that if all our eigenvectors come from distinct eigenvalues, linear independence is guaranteed, and we automatically have a fundamental solution set.

**Corollary 3.** If the $n \times n$ constant matrix $A$ has $n$ distinct eigenvalues $r_1, \ldots, r_n$ and $u_i$ is the eigenvector associated with $r_i$, then $\{e^{r_1 t}u_1, \ldots, e^{r_n t}u_n\}$ is a fundamental solution set for the homogeneous system $x' = Ax$.

**Example 4.** Solve the IVP $x'(t) = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} x(t)$, $x(0) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$.

**Solution.** In Example 2, we showed that the coefficient matrix had three distinct eigenvalues $r_1 = 1, r_2 = 2, r_3 = 3$ with corresponding eigenvectors $u_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}, u_3 = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ (by taking $s = 1$). Since the eigenvalues are distinct, the linear independence of the eigenvectors is assured, so a general solution is
\[
\begin{align*}
x(t) &= c_1 e^t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \\
&= e^t \begin{pmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.
\end{align*}
\]

To satisfy the initial condition, substitute $t = 0$ to get
\[
\begin{pmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix},
\]
which implies $c_1 = 0, c_2 = 1, c_3 = -1$. Therefore, the solution is
\[
x(t) = e^{2t} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - e^{3t} \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}. \quad \diamondsuit
\]
**Definition.** A real symmetric matrix $A$ is a matrix with real entries that satisfies $A^T = A$.

Recall that taking the transpose of a matrix simply interchanges its rows and columns; therefore, the entries of a symmetric matrix are symmetric about its main diagonal (hence the name). We mention this definition because it turns out that if $A$ is an $n \times n$ real symmetric matrix, then there **always** exist $n$ linearly independent eigenvectors, even if they do not all come from distinct eigenvalues.

**Example 5.** Find a general solution of $x'(t) = Ax(t)$, where $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

**Solution.** Since $A$ is symmetric, we are guaranteed to have three linearly independent eigenvectors. We find the characteristic polynomial:

$$|A - rI| = \begin{vmatrix} 1 - r & -2 & 2 \\ -2 & 1 - r & 2 \\ 2 & 2 & 1 - r \end{vmatrix} = -(r - 3)^2(r + 3) = 0.$$  

Therefore, the eigenvalues are $r_1 = 3, r_2 = -3$. Since $r_1$ has multiplicity two as a root of the characteristic polynomial, we must find two linearly independent eigenvectors associated with $r_1 = 3$. We solve

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which is equivalent to the single equation $u_1 = u_3 - u_2$. Assigning arbitrary values to both $u_2$ and $u_3$, say $u_2 = v, u_3 = s$, then $u_1 = s - v$ and the eigenvector is

$$u = \begin{bmatrix} s - v \\ v \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$  

By taking $(s, v) = (1, 0)$ and $(s, v) = (0, 1)$ we get two linearly independent eigenvectors $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

For $r_2 = -3$, we solve

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
Letting $u_3$ take the value 1, we get the eigenvector $u_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. Then a general solution is given by

$$x(t) = c_1e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Note that if a matrix $A$ is not symmetric, it is possible for $A$ to have a repeated eigenvalue which does not generate multiple linearly independent eigenvectors (consider for example $A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$); we defer the theory for finding a general solution in this case to Section 9.8.

**Remark.** If an $n \times n$ matrix $A$ has $n$ linearly independent eigenvectors, we say that $A$ is **diagonalizable**; this is because it can be written as $A = UDU^{-1}$ where $U$ is the matrix whose columns are the eigenvectors of $A$ and $D$ is a diagonal matrix whose diagonal entries are the eigenvalues of $A$.