Section 9.5: Homogeneous Linear Systems with Constant Coefficients

Eigenvalues and Eigenvectors

We now show how to obtain a general solution for the homogeneous system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) \tag{1}$$

where $\mathbf{A}$ is a constant $n \times n$ matrix. Recall from the previous section that this amounts to finding n linearly independent solutions to (1). In Chapter 4, we used the fact that homogeneous linear equations with constant coefficients had solutions of the form e^{rt}; extending this idea, we guess that the system (1) will have solutions of the form $\mathbf{x}(t) = \mathbf{u}e^{rt}$ for some constant r and constant vector $\mathbf{u}$. Indeed, substituting this vector into (1) gives

$$re^{rt}\mathbf{u} = \mathbf{A}e^{rt}\mathbf{u} = e^{rt}\mathbf{A}\mathbf{u}.$$

Rearranging terms after canceling the exponential yields

$$(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{0}, \tag{2}$$

where $\mathbf{I}$ denotes the identity matrix.

Therefore, $\mathbf{x}(t) = e^{rt}\mathbf{u}$ is a solution to (1) if and only if r and $\mathbf{u}$ satisfy equation (2). This equation is trivially satisfied when $\mathbf{u} = \mathbf{0}$, but this is not part of any linearly independent set (why?), so we require also that $\mathbf{u} \neq \mathbf{0}$. There is a special name for such r and $\mathbf{u}$.

Definition. Let $\mathbf{A}$ be an $n \times n$ constant matrix. The eigenvalues of $\mathbf{A}$ are those (real or complex) numbers r for which $(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{0}$ has at least one nontrivial solution $\mathbf{u}$. The corresponding nontrivial solutions $\mathbf{u}$ are called the eigenvectors of $\mathbf{A}$ associated with r.

A basic fact of linear algebra (mentioned in Section 9.3) is that (2) will have a nontrivial solution if and only if the determinant $|\mathbf{A} - r\mathbf{I}| = 0$. Since the determinant of this matrix is a polynomial in r of degree n, call it $p(r)$, to find the eigenvalues of a matrix $\mathbf{A}$ we must find the zeros of the **characteristic polynomial** $p(r)$. This is similar to the auxiliary equation for scalar DE's.

Example 1. Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

Solution. We find the characteristic polynomial for $\mathbf{A}$:

$$|\mathbf{A} - r\mathbf{I}| = \begin{vmatrix} 2-r & -3 \\ 1 & -2-r \end{vmatrix} = (2-r)(-2-r) + 3 = r^2 - 1 = 0.$$

Therefore, the eigenvalues of $\mathbf{A}$ are $r_1 = 1, r_2 = -1$. For the eigenvectors corresponding to $r_1 = 1$, we solve the equation $(\mathbf{A} - \mathbf{I})\mathbf{u} = \mathbf{0}$:

$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is the equation $u_1 = 3u_2$, so if we set $u_2 = s$, then $\mathbf{u}_1 = s\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

For $r_2 = -1$, we solve $(\mathbf{A} + \mathbf{I})\mathbf{u} = \mathbf{0}$:

$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is the equation $u_1 = u_2$, so if we set $u_2 = s$, then $\mathbf{u}_2 = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. ◊

Remark. The set of eigenvectors for r_1 forms a subspace of $\mathbb{R}^2$ when the zero vector is adjoined (and likewise for the set of eigenvectors for r_2). These subspaces are called **eigenspaces**.

Example 2. Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$.

Solution. We find the characteristic polynomial:

$$|\mathbf{A} - r\mathbf{I}| = \begin{vmatrix} 1-r & 2 & -1 \\ 1 & -r & 1 \\ 4 & -4 & 5-r \end{vmatrix} = (r-1)(r-2)(r-3) = 0.$$

Therefore, the eigenvalues of $\mathbf{A}$ are $r_1 = 1, r_2 = 2, r_3 = 3$. To find eigenvectors for $r_1 = 1$, solve $(\mathbf{A} - \mathbf{I})\mathbf{u} = \mathbf{0}$:

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the third row is a multiple of the second row, this is equivalent to the system

$$\begin{aligned} u_1 - u_2 + u_3 &= 0, \\ 2u_2 - u_3 &= 0. \end{aligned}$$

If we assign an arbitrary value s to u_2, solving the second equation gives $u_3 = 2s$, and substituting these into the first equation gives $u_1 = -s$. Therefore, the eigenvectors for r_1 are $\mathbf{u}_1 = s \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

For $r_2 = 2$, we solve

$$\begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now the first and second equations are multiples, so an equivalent system is

$$\begin{aligned} u_1 - 2u_2 + u_3 &= 0, \\ 4u_1 - 4u_2 + 3u_3 &= 0. \end{aligned}$$

Again letting $u_2 = s$ gives $u_3 = 4s, u_1 = -2s$, so the eigenvectors for r_2 are $\mathbf{u}_2 = s \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$.

Finally, for $r_3 = 3$ we solve

$$\begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first and third equations are multiples, so an equivalent system is

$$\begin{aligned} u_1 - 3u_2 + u_3 &= 0, \\ 2u_1 - 2u_2 + u_3 &= 0. \end{aligned}$$

Letting $u_2 = s$ gives $u_3 = 4s, u_1 = -s$, so the eigenvectors for r_3 are $\mathbf{u}_3 = s \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$. ♢

The Case of n Linearly Independent Eigenvectors

How does this calculation help us with the general solution to our DE? If our matrix has n linearly independent eigenvectors, then we will have enough solutions to write the general solution, as stated in the following theorem.

Theorem 1. *Suppose the $n \times n$ constant matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Let r_i be the eigenvalue corresponding to $\mathbf{u}_i$. Then $\{e^{r_1 t}\mathbf{u}_1, e^{r_2 t}\mathbf{u}_2, \ldots, e^{r_n t}\mathbf{u}_n\}$ is a fundamental solution set on $(-\infty, \infty)$ for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, which has general solution*

$$\mathbf{x}(t) = c_1 e^{r_1 t}\mathbf{u}_1 + c_2 e^{r_2 t}\mathbf{u}_2 + \cdots + c_n e^{r_n t}\mathbf{u}_n, \tag{3}$$

where $c_1, \ldots, c_n$ are arbitrary constants.

Example 3. Find a general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

Solution. In Example 1, we found that this matrix has eigenvalues $r_1 = 1, r_2 = -1$ with eigenvectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where we have taken $s = 1$. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent, a general solution is

$$\mathbf{x}(t) = c_1 e^{t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ ♢

The next theorem is very useful for confirming linear independence of eigenvectors.

Theorem 2. *If $r_1, \ldots, r_m$ are distinct eigenvalues for the matrix $\mathbf{A}$ and $\mathbf{u}_i$ is an eigenvector associated with r_i, then $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent.*

Proof. We only prove the case $m = 2$; the more general result follows by induction. Suppose by way of contradiction that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly dependent; that is, for some constant c we have

$$\mathbf{u}_1 = c\mathbf{u}_2. \tag{4}$$

Multiplying (4) through by $\mathbf{A}$, we have

(5) $$\mathbf{A}\mathbf{u}_1 = c\mathbf{A}\mathbf{u}_2 \Rightarrow r_1\mathbf{u}_1 = cr_2\mathbf{u}_2.$$

Multiplying (4) by r_2 and subtracting this from (5) gives

$$(r_1 - r_2)\mathbf{u}_1 = \mathbf{0}.$$

Since $\mathbf{u}_1 \neq \mathbf{0}$, this implies $r_1 = r_2$, but this contradicts the assumption that the eigenvalues were distinct. □

This means that if all our eigenvectors come from distinct eigenvalues, linear independence is guaranteed, and we automatically have a fundamental solution set.

Corollary 3. *If the $n \times n$ constant matrix $\mathbf{A}$ has n distinct eigenvalues $r_1, \ldots, r_n$ and $\mathbf{u}_i$ is the eigenvector associated with r_i, then $\{e^{r_1 t}\mathbf{u}_1, \ldots, e^{r_n t}\mathbf{u}_n\}$ is a fundamental solution set for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.*

Example 4. Solve the IVP $\mathbf{x}'(t) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$

Solution. In Example 2, we showed that the coefficient matrix had three distinct eigenvalues $r_1 = 1, r_2 = 2, r_3 = 3$ with corresponding eigenvectors $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ (by taking $s = 1$). Since the eigenvalues are distinct, the linear independence of the eigenvectors is assured, so a general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -e^t & -2e^{2t} & -e^{3t} \\ e^t & e^{2t} & e^{3t} \\ 2e^t & 4e^{2t} & 4e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \end{aligned}$$

To satisfy the initial condition, substitute $t = 0$ to get

$$\begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

which implies $c_1 = 0, c_2 = 1, c_3 = -1$. Therefore, the solution is

$$\mathbf{x}(t) = e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} - e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}.$$ ◊

Definition. A real symmetric matrix $\mathbf{A}$ is a matrix with real entries that satisfies $\mathbf{A}^T = \mathbf{A}$.

Recall that taking the transpose of a matrix simply interchanges its rows and columns; therefore, the entries of a symmetric matrix are symmetric about its main diagonal (hence the name). We mention this definition because it turns out that if $\mathbf{A}$ is an $n \times n$ real symmetric matrix, then there *always* exist n linearly independent eigenvectors, *even if they do not all come from distinct eigenvalues.*

Example 5. Find a general solution of $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

Solution. Since $\mathbf{A}$ is symmetric, we are guaranteed to have three linearly independent eigenvectors. We find the characteristic polynomial:

$$|\mathbf{A} - r\mathbf{I}| = \begin{vmatrix} 1-r & -2 & 2 \\ -2 & 1-r & 2 \\ 2 & 2 & 1-r \end{vmatrix} = -(r-3)^2(r+3) = 0.$$

Therefore, the eigenvalues are $r_1 = 3, r_2 = -3$. Since r_1 has multiplicity two as a root of the characteristic polynomial, we must find two linearly independent eigenvectors associated with $r_1 = 3$. We solve

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which is equivalent to the single equation $u_1 = u_3 - u_2$. Assigning arbitrary values to both u_2 and u_3, say $u_2 = v, u_3 = s$, then $u_1 = s - v$ and the eigenvector is

$$\mathbf{u} = \begin{bmatrix} s-v \\ v \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

By taking $(s, v) = (1, 0)$ and $(s, v) = (0, 1)$ we get two linearly independent eigenvectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

For $r_2 = -3$, we solve

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Letting u_3 take the value 1, we get the eigenvector $\mathbf{u}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. Then a general solution is given by

$$\mathbf{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}. \qquad \diamond$$

Note that if a matrix $\mathbf{A}$ is not symmetric, it is possible for $\mathbf{A}$ to have a repeated eigenvalue which does not generate multiple linearly independent eigenvectors (consider for example $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$); we defer the theory for finding a general solution in this case to Section 9.8.

Remark. If an $n \times n$ matrix $\mathbf{A}$ has n linearly independent eigenvectors, we say that $\mathbf{A}$ is **diagonalizable**; this is because it can be written as $\mathbf{A} = \mathbf{UDU}^{-1}$ where $\mathbf{U}$ is the matrix whose columns are the eigenvectors of $\mathbf{A}$ and $\mathbf{D}$ is a diagonal matrix whose diagonal entries are the eigenvalues of $\mathbf{A}$.

Homework: pp. 534-535 #1-7 odd, 11-15 odd, 19-23 odd, 31, 33.