MAP 2302

Section 4787

Exam 1

July 8, 2016

Name: Solutions

This exam consists of 8 free-response problems. There are 52 points possible, but the exam is only counted out of 50 points (so it is possible to get 104%).

You may not use any books, notes, or calculators on this exam.

You are required to show sufficient work for each problem on the exam. For the sake of your instructor (grader), please abide by the following guidelines:

- Organize your work in a neat and coherent way in the space provided. Please try not to scatter work all over the page in a complicated fashion. You may want to use scratch paper when first solving the problem, and then neatly copy your relevant and organized work onto the test.
- Answers which are not justified will not receive full credit. You must show sufficient work to indicate you understand all the steps involved in solving the problem. A correct answer with no supporting work will receive little, if any, credit, but an incorrect final answer with accompanying work will receive credit proportional to the accuracy of the work.
- If you need more space for a problem, use a (clean) piece of scratch paper and clearly indicate which problem you are solving on the page, as well as notifying your instructor of this when you turn in the exam.

You will have 75 minutes to complete the exam.

Your signature below indicates that you promise to abide by the UF Honor Code.

I have neither given nor received unauthorized help on this exam.

Signature ______________________________

Do your best, and good luck!

1. (6 points) Find an explicit solution to the initial value problem

$$\frac{dy}{dx} = (1+y^2)\tan x, \quad y(0) = \sqrt{3}.$$

Separating variables gives $\int \frac{dy}{1+y^2} = \int \tan x \, dx$

$\Rightarrow \arctan y = -\ln|\cos x| + C$

$\Rightarrow y = \tan(-\ln|\cos x| + C)$ (General soln)

I.C. : $y(0) = \tan(-\ln|\cos 0| + C) = \tan C = \sqrt{3}$

$\Rightarrow C = \arctan(\sqrt{3}) = \frac{\pi}{3}$

Thus, $\boxed{y = \tan\left(-\ln|\cos x| + \frac{\pi}{3}\right)}$

$\left[\int \tan x dx = \int \frac{\sin x \, dx}{\cos x} = -\int \frac{du}{u} = -\ln|u| = -\ln|\cos x|\right]$

$u = \cos x, \; du = -\sin x dx$

2. (5 points) Use implicit differentiation to show that the relation $x^4 + \cos(x^2 y) = 5$ is an implicit solution to the differential equation

$$\frac{dy}{dx} = 4x\csc(x^2 y) - 2yx^{-1}.$$

$\frac{d}{dx}\left[x^4 + \cos(x^2 y) = 5\right]$

$4x^3 - \sin(x^2 y)(2xy + x^2 \frac{dy}{dx}) = 0$

$\Rightarrow 2xy + x^2 \frac{dy}{dx} = \frac{4x^3}{\sin(x^2 y)}$

$\Rightarrow x^2 \frac{dy}{dx} = 4x^3 \csc(x^2 y) - 2xy$

$\Rightarrow \frac{dy}{dx} = 4x\csc(x^2 y) - 2x^{-1}y.$ ✓

3. (8 points) Find an explicit general solution to the following Bernoulli equation:

$$\frac{dy}{dx} + y = e^x y^{-2}.$$

Divide by y^{-2} / multiply by y^2 :

$$y^2 \frac{dy}{dx} + y^3 = e^x$$

Let $v = y^{1-(-2)} = y^3 \Rightarrow \frac{dv}{dx} = 3y^2 \frac{dy}{dx}$. Then

$$\frac{1}{3} \cdot \frac{dv}{dx} + v = e^x$$

$$\Rightarrow \frac{dv}{dx} + 3v = 3e^x.$$

I.F. $\mu(x) = e^{\int 3dx} = e^{3x}$, so

$$e^{3x} \frac{dv}{dx} + 3ve^{3x} = 3e^{4x}$$

$$\Rightarrow \int \frac{d}{dx}\left[e^{3x} v\right] dx = \int 3e^{4x}\, dx$$

$$\Rightarrow e^{3x} v = \frac{3}{4} e^{4x} + C$$

$$\Rightarrow v = y^3 = \frac{3}{4} e^x + Ce^{-3x}$$

$$\Rightarrow \boxed{y = \left(\frac{3}{4} e^x + Ce^{-3x}\right)^{1/3}}$$

4. (a) (4 points) Sketch the direction field of the equation $\frac{dy}{dx} = -\frac{y}{x}$ by finding isoclines for the values $C = \pm 1, \pm 2$.

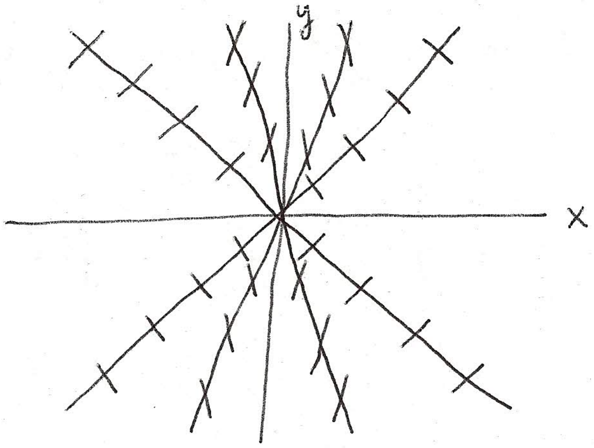
$-\frac{y}{x} = C \Rightarrow y = -Cx$

$C = 1: \quad y = -x$

$C = -1: \quad y = x$

$C = 2: \quad y = -2x$

$C = -2: \quad y = 2x$



(b) (1 point) If $y_1(x)$ is the solution curve satisfying the initial condition $y(1) = 3$, what is $\lim_{x\to\infty} y_1(x)$?

0

(c) (1 point) If $y_2(x)$ is the solution curve satisfying the initial condition $y(-2) = -1$, what is the limit of the y-values of $y_2(x)$ as x increases?

$-\infty$

5. (4 points) Use Euler's method with step size $h = 0.5$ to approximate the solution to the initial value problem

$$y' = x + y, \; y(0) = 1$$

at the point $x = 1$.

$\phi(x_{n+1}) \approx y_{n+1} = y_n + hf(x_n, y_n) = y_n + \frac{1}{2}(x_n + y_n)$

$(x_0, y_0) = (0, 1)$

$\phi(0.5) \approx y_1 = y_0 + \frac{1}{2}(x_0 + y_0) = 1 + \frac{1}{2}(1) = \frac{3}{2}$

$\phi(1) \approx y_2 = y_1 + \frac{1}{2}(x_1 + y_1) = \frac{3}{2} + \frac{1}{2}(2) = \frac{5}{2} = 2.5$

6. (6 points) Find an explicit general solution to the equation

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x\sin(2x).$$

Standard form: $\frac{dy}{dx} - \frac{y}{x} = x^2 \sin(2x)$

I.F. : $\mu(x) = e^{\int p(x)dx} = e^{\int -1/x\, dx} = e^{-\ln x} = \underline{x^{-1}}$;

Multiply by IF : $\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x\sin(2x)$ ← Could start here

$$\int \frac{d}{dx}\left[x^{-1}\cdot y\right] = \int x \sin(2x)\, dx$$

IBP

x	$+$	$\sin(2x)$
1	$-$	$-\frac{1}{2}\cos(2x)$
0		$-\frac{1}{4}\sin(2x)$

$$x^{-1}y = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C$$

$$\Rightarrow \boxed{y = -\frac{1}{2}x^2\cos(2x) + \frac{1}{4}x\sin(2x) + Cx}$$

7. (7 points) Find an explicit general solution to the homogeneous equation

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2}.$$

$$\frac{dy}{dx} = \frac{xy}{x^2} + \frac{y^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

Let $v = \frac{y}{x}$ $\Rightarrow$ $v + x\frac{dv}{dx} = v + v^2$

$$\Rightarrow x\frac{dv}{dx} = v^2$$

$$\Rightarrow \int v^{-2}\, dv = \int \frac{dx}{x}$$

$$\Rightarrow -v^{-1} = \ln|x| + C$$

$$\Rightarrow v = \frac{y}{x} = \frac{-1}{\ln|x| + C}$$

$$\Rightarrow \boxed{y = \frac{-x}{\ln|x| + C}}$$

8. (10 points) Find an integrating factor and use it to solve the following equation:

$$(xy - y^2 - 2y)dx + (xy - x^2 + 4x)dy = 0.$$

$$M_y = \frac{\partial}{\partial y}\left[xy - y^2 - 2y\right] = x - 2y - 2$$

$\Rightarrow$ not exact.

$$N_x = \frac{\partial}{\partial x}\left[xy - x^2 + 4x\right] = y - 2x + 4$$

$$\frac{N_x - M_y}{M} = \frac{(y-2x+4)-(x-2y-2)}{xy-y^2-2y} = \frac{3y-3x+6}{xy-y^2-2y} = \frac{-3(x-y-2)}{y(x-y-2)}$$

$$= -\frac{3}{y}$$ depends only on y

I.F. $\mu(y) = e^{\int -3/y\,dy} = e^{-3\ln y} = y^{-3}$

Multiplying by IF gives

$$(xy^{-2} - y^{-1} - 2y^{-2})dx + (xy^{-2} - x^2y^{-3} + 4xy^{-3})dy = 0$$

Now $M_y = -2xy^{-3} + y^{-2} + 4y^{-3}$, $N_x = y^{-2} - 2xy^{-3} + 4y^{-3}$

$\Rightarrow$ exact.

$$\int M\,dx = \int (xy^{-2} - y^{-1} - 2y^{-2})\,dx = \frac{1}{2}x^2y^{-2} - xy^{-1} - 2xy^{-2} + g(y)$$

$$\int N\,dy = \int (xy^{-2} - x^2y^{-3} + 4xy^{-3})\,dy = -xy^{-1} + \frac{1}{2}x^2y^{-2} - 2xy^{-2} + h(x)$$

$$\Rightarrow \boxed{\frac{1}{2}x^2y^{-2} - xy^{-1} - 2xy^{-2} = C}$$