Exam 2 Solutions

① Let G + H be groups. An <u>isomorphism</u> from G to H is a map $\phi : G \to H$ such that ϕ is a bijection, and for every $g_1, g_2 \in G$, we have

$$\phi(g_1 g_2) = \phi(g_1)\,\phi(g_2).$$

The groups $U(3)$ and $\mathbb{Z}_2$ are isomorphic.

The groups $\mathbb{Z}_8$ and D_4 both have order 8, but are not isomorphic since $\mathbb{Z}_8$ is cyclic and D_4 is not cyclic. (Many other examples are possible.)

② Let G be a group and $H \leq G$. A <u>left coset</u> of H in G is the set $aH = \{ah \mid h \in H\}$ for some $a \in G$.

If $a, b \in G$, a necessary and sufficient condition for $aH = bH$ is one of the following: $a \in bH$, $b \in aH$, $b^{-1}a \in H$, $a^{-1}b \in H$.

③
$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 7 & 11 & 5 & 10 & 6 & 12 & 4 & 2 & 1 & 3 & 9 \end{bmatrix}$$

$$\Rightarrow \sigma = (1, 8, 4, 5, 10)(2, 7, 12, 9)(3, 11)$$

Then $|\sigma| = \text{lcm}(5, 4, 2) = \boxed{20}$.

Since $\sigma = (1,10)(1,5)(1,4)(1,8)(2,9)(2,12)(2,7)(3,11)$ can be written as a product of an even number (8) of transpositions, σ is $\boxed{\text{even}}$.

④ Let G be a group. Prove that $\phi(g) = g^{-1}$ is an automorphism iff G is abelian.

<u>Proof</u> Suppose first that ϕ is an automorphism. Let $a, b \in G$. Then $\phi(ab) = (ab)^{-1} = b^{-1}a^{-1}$ and $\phi(a)\phi(b) = a^{-1}b^{-1}$. Since ϕ preserves the group operation, $\phi(ab) = \phi(a)\phi(b) \Rightarrow b^{-1}a^{-1} = a^{-1}b^{-1} \Rightarrow (b^{-1}a^{-1})^{-1} = (a^{-1}b^{-1})^{-1}$ $\Rightarrow ab = ba$. Thus, G is abelian.

Suppose now that G is abelian. Let $x, y \in G$ and suppose $\phi(x) = \phi(y)$. Then $x^{-1} = y^{-1} \Rightarrow (x^{-1})^{-1} = (y^{-1})^{-1} \Rightarrow x = y$, so ϕ is injective.

If $x \in G$, then $x^{-1} \in G$ and $\phi(x^{-1}) = (x^{-1})^{-1} = x$, so ϕ is surjective.

Finally, let $x, y \in G$. Then $\phi(xy) = \phi(yx) = (yx)^{-1} = x^{-1}y^{-1} = \phi(x)\phi(y)$, so ϕ preserves the group operation. Thus, ϕ is an automorphism. ▨

(5) Does the group A_7 have any elements of order 6?

Yes. Consider $\sigma = (12)(34)(567)$. We have $|\sigma| = \operatorname{lcm}(2, 2, 3) = 6$.

Also, $\sigma = (12)(34)(56)(67)$ is the product of an even number (4) of transpositions, so $\sigma \in A_7$.

(6) Let G be a group, $|G| = 30$, and $H \leq G$. If H is not cyclic and $|G:H|$ is even, what is $|H|$?

By Lagrange's Thrm, we Know that $|H|$ divides $|G|$, so the possibilities for $|H|$ are $\{1, 2, 3, 5, 6, 10, 15, 30\}$. If H had prime order, then by a corollary to Lagrange's Thrm it would be cyclic. Therefore, $|H|$ is not prime. For the same reason, $|H| \neq 1$, so the remaining possibilities are $\{6, 10, 15, 30\}$. Lastly, by Lagrange's Thrm again we Know $|G:H| = \frac{|G|}{|H|} = \frac{30}{|H|}$. Calculating $|G:H|$ for the remaining options yields $\frac{30}{6} = 5$, $\frac{30}{10} = 3$, $\frac{30}{15} = 2$, and $\frac{30}{30} = 1$. Only one of these numbers is even, so we must have $|H| = \boxed{15}$.