MAP 2302

Section 4787

Exam 2

July 22, 2016

Name: Solutions

This exam consists of 7 free-response problems. There are 52 points possible, but the exam is only counted out of 50 points (so it is possible to get 104%).

You may not use any books, notes, or calculators on this exam.

You are required to show sufficient work for each problem on the exam. For the sake of your instructor (grader), please abide by the following guidelines:

- Organize your work in a neat and coherent way in the space provided. Please try not to scatter work all over the page in a complicated fashion. You may want to use scratch paper when first solving the problem, and then neatly copy your relevant and organized work onto the test.
- Answers which are not justified will not receive full credit. You must show sufficient work to indicate you understand all the steps involved in solving the problem. A correct answer with no supporting work will receive little, if any, credit, but an incorrect final answer with accompanying work will receive credit proportional to the accuracy of the work.
- If you need more space for a problem, use a (clean) piece of scratch paper and clearly indicate which problem you are solving on the page, as well as notifying your instructor of this when you turn in the exam.

You will have 75 minutes to complete the exam.

Your signature below indicates that you promise to abide by the UF Honor Code.

I have neither given nor received unauthorized help on this exam.

Signature ________________________________

Do your best, and good luck!

1. (4 points) Solve the initial value problem
$$y'' - 4y' + 4y = 0,\ y(0) = 1,\ y'(0) = 1.$$

Aux. eqn.: $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$ (mult. 2)

General solution: $y = c_1 e^{2t} + c_2 t e^{2t}$

$y' = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}$

I.C. $y(0) = c_1 = 1$

$y'(0) = 2c_1 + c_2 = 1 \Rightarrow 2 + c_2 = 1 \Rightarrow c_2 = -1$

Solution to IVP is $\boxed{y(t) = e^{2t} - te^{2t}}$

2. (5 points) Find a general solution for the differential equation
$$y''' - 3y'' + 9y' + 13y = 0.$$

Aux eqn.: $r^3 - 3r^2 + 9r + 13 = 0$

Try $r = -1$:

$$\begin{array}{r|rrr|r} -1 & 1 & -3 & 9 & 13 \\ + & 0 & -1 & 4 & -13 \\ \hline & 1 & -4 & 13 & 0 \end{array}$$

$\Rightarrow (r+1)(r^2 - 4r + 13) = 0$

$\Rightarrow r = -1,\ r = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm \sqrt{-36}}{2} = \dfrac{4 \pm 6i}{2} = 2 \pm 3i$

General solution: $\boxed{y(t) = c_1 e^{-t} + c_2 e^{2t}\cos(3t) + c_3 e^{2t}\sin(3t)}$

3. Consider the nonhomogeneous differential equation

$$xy''' - y'' = -2.$$

A particular solution to the equation is $y_p = x^2$, and a fundamental solution set for the corresponding homogeneous equation is $S = \{1, x, x^3\}$.

(a) (2 points) Verify that the functions in S are linearly independent on $(0, \infty)$ by calculating the Wronskian.

$$W[y_1, y_2, y_3](x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = 1\cdot(6x-0) - x(0-0) + x^3(0-0)$$

$$= 6x$$

Since $6x \neq 0$ on $(0, \infty)$, the functions are linearly independent.

(b) (1 point) Write a general solution to the nonhomogeneous equation.

$$y = y_h + y_p = c_1 + c_2 x + c_3 x^3 + x^2$$

(c) (2 points) Given that $y_p = 2x^4$ is a particular solution to the equation

$$xy''' - y'' = 24x^2$$

find a general solution to the equation

$$xy''' - y'' = 12x^2 - 6.$$

Since $12x^2 - 6 = \frac{1}{2} g_1 + 3g_2$, where $g_1 = 24x^2$ and $g_2 = -2$,

and $y_{p_1}(x) = 2x^4$ solves $xy''' - y'' = g_1 = 24x^2$ and

$y_{p_2}(x) = x^2$ solves $xy''' - y'' = g_2 = -2$,

by the superposition principle, a general solution is

$$y = \frac{1}{2} y_{p_1} + 3y_{p_2} + y_h$$

$$\boxed{y(x) = x^4 + 3x^2 + c_1 + c_2 x + c_3 x^3}$$

4. (10 points) Use the method of undetermined coefficients to find a general solution to the differential equation

$$y'' + 16y = 34te^t - 13e^t.$$

Aux eqn.: $r^2 + 16 = 0 \Rightarrow r^2 = -16 \Rightarrow r = \pm 4i$

Then $y_h = c_1 \cos(4t) + c_2 \sin(4t)$.

Guess $y_p = (At + B)e^t$

$y_p' = Ae^t + (At + B)e^t$

$y_p'' = 2Ae^t + (At + B)e^t$

Substitute into DE:

$$\begin{aligned} y_p'' + 16y_p &= Ate^t + (2A + B)e^t \\ &\quad + 16Ate^t + 16Be^t \\ \hline &= 17Ate^t + (2A + 17B)e^t = 34te^t - 13e^t \end{aligned}$$

$$\Rightarrow \left| \begin{aligned} &17A = 34 && \Rightarrow A = 2 \\ &2A + 17B = -13 && \Rightarrow 4 + 17B = -13 \Rightarrow 17B = -17 \Rightarrow B = -1 \end{aligned} \right.$$

So $y_p = (2t - 1)e^t$ and a general solution is

$$\boxed{y(t) = c_1 \cos(4t) + c_2 \sin(4t) + (2t - 1)e^t}$$

5. (6 points) Suppose that a homogeneous linear equation with constant coefficients has the auxiliary equation

$$(r-4)(r-(-1+2i))^2(r-(-1-2i))^2 = 0.$$

(a) Write the general solution to the differential equation.

$$y(t) = c_1 e^{4t} + c_2 e^{-t}\cos(2t) + c_3 t e^{-t}\cos(2t) + c_4 e^{-t}\sin(2t) + c_5 t e^{-t}\sin(2t)$$

(b) What is the order of the equation?

5^{th} order

6. (9 points) Consider the differential equation

$$y'' + 6y' + 9y = f(t).$$

Aux eqn: $r^2+6r+9=0$
$(r+3)^2=0 \Rightarrow r=-3$ is a double root

For each of the nonhomogeneous terms below, write the form of a particular solution to the equation suggested by the method of undetermined coefficients. DO NOT find the coefficients.

(a) $f(t) = t^2 e^{3t}$

$$y_p = (At^2+Bt+C)e^{3t}$$

Since $r=3$ is not a root, take $s=0$.

(b) $f(t) = te^{-3t}\cos(3t)$

$$y_p = (A_1 t + A_0)e^{-3t}\cos(3t) + (B_1 t + B_0)e^{-3t}\sin(3t)$$

Since $-3+3i$ is not a root, take $s=0$.

(c) $f(t) = (t+2)e^{-3t}$

Since $r=-3$ is a double root, take $s=2$.

$$y_p = t^2(At+B)e^{-3t}$$

7. Consider the differential equation
$$t^2y'' + 3ty' + y = t^{-1} \quad (t > 0).$$

(a) (2 points) Find the general solution to the corresponding homogeneous equation.

Characteristic eqn: $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$ (mult. 2)

Gen. soln.: $\boxed{y(t) = c_1 t^{-1} + c_2 t^{-1} \ln t}$

(b) (2 points) Calculate the Wronskian of the homogeneous solutions.

$$W[y_1, y_2] = \begin{vmatrix} t^{-1} & t^{-1}\ln t \\ -t^{-2} & -t^{-2}\ln t + t^{-2} \end{vmatrix} = -t^{-3}\ln t + t^{-3} + t^{-3}\ln t$$
$$= \boxed{t^{-3}}$$

(c) (8 points) Use variation of parameters to find a particular solution to the nonhomogeneous equation. $g = t^{-1}/t^2 = t^{-3}$

$$v_1 = -\int \frac{g y_2}{W[y_1, y_2]} = -\int \frac{t^{-3} \cdot t^{-1}\ln t}{t^{-3}}\,dt = -\int \frac{\ln t}{t}\,dt$$

$u = \ln t,\ du = \frac{1}{t}dt$ $\quad = -\int u\,du = -\frac{1}{2}u^2 = -\frac{1}{2}(\ln t)^2$

$$v_2 = \int \frac{g y_1}{W[y_1, y_2]} = \int \frac{t^{-3} \cdot t^{-1}}{t^{-3}}\,dt = \int \frac{dt}{t} = \ln t$$

$$y_p = v_1 y_1 + v_2 y_2 = -\frac{1}{2}t^{-1}(\ln t)^2 + t^{-1}(\ln t)^2$$
$$= \boxed{\frac{1}{2}t^{-1}(\ln t)^2}$$

(d) (1 point) Write the general solution to the nonhomogeneous equation.

$$y = y_h + y_p = c_1 t^{-1} + c_2 t^{-1}\ln t + \frac{1}{2}t^{-1}(\ln t)^2$$