Abstract Algebra 1

MAS 4301

Section 8385

Exam 3

July 14, 2017

Please answer the following questions on the scratch paper provided. Be sure to put your name at the top of each page of your work. You may not use any books, notes, or calculators on this exam. You are required to show sufficient work for each question on the exam in order to receive full credit. You may use results we proved in class as long as you make it clear that you are doing so and you state them clearly.

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1. (7 points) Let G be a group and $H \leq G$. Define what it means for H to be *normal* in G. State a necessary and sufficient condition for $H \trianglelefteq G$. Give an example of a nontrivial proper normal subgroup of the dihedral group D_4, and briefly explain why it is normal (without using the definition).

2. (7 points) Let G be a group and H, K be subgroups of G. State the conditions under which G is the *internal direct product* of H and K. Is $S_3 = \langle (123) \rangle \times \langle (12) \rangle$? Explain why or why not.

3. (9 points) Let $G = \mathbb{Z}_{15} \oplus \mathbb{Z}_{20}$. Determine the number of elements of order 10 and the number of cyclic subgroups of order 10 in G.

4. (8 points) Let G be an abelian group and $n \in \mathbb{Z}^+$. Define the subgroups
$H = \{g \in G \mid g^n = e\}$ and $K = \{g^n \mid g \in G\}$. Prove that $G/H \cong K$.

5. (9 points) (a) How many homomorphisms are there $\phi : \mathbb{Z}_{45} \to \mathbb{Z}_{30}$?
(b) Suppose that $\phi(\overline{1}) = \overline{2}$. Find $\phi^{-1}(\overline{8})$ (list the elements explicitly).

6. (10 points) Classify all abelian groups of order 540. If G is an abelian group of order 540, G has no element of order 27, has an element of order 9, and has three elements of order 2, what is the isomorphism class of G? Justify your choice.