Exam 3 Solutions

1. Let H be a subgroup of G. H is <u>normal</u> in G if $aH = Ha \quad \forall a \in G$.

A necessary and sufficient condition for $H \trianglelefteq G$ is $xHx^{-1} \subseteq H \quad \forall x \in G$.

The subgroup $\{R_0, R_{180}\}$ is a nontrivial proper normal subgroup of D_4 because $Z(G) = \{R_0, R_{180}\}$ and $Z(G) \trianglelefteq G$ for every group G.

The subgroups $\{R_0, R_{180}, H, V\}$, $\{R_0, R_{180}, D, D'\}$, and $\{R_0, R_{90}, R_{180}, R_{270}\}$ are all normal because they have index 2 in D_4.

2. Let $H, K \leq G$. G is the <u>internal direct product</u> of H and K if $H \trianglelefteq G$, $K \trianglelefteq G$, $G = HK$, and $H \cap K = \{e_G\}$.

$S_3 \neq \langle (123) \rangle \times \langle (12) \rangle$ because $\langle (12) \rangle$ is not normal in S_3.

$[\,(13)(12) \neq (12)(13) \Rightarrow (13)\langle (12) \rangle \neq \langle (12) \rangle (13)\,]$

3. $G = \mathbb{Z}_{15} \oplus \mathbb{Z}_{20}$. The element (a, b) has order 10 if $\operatorname{lcm}(|a|, |b|) = 10$.

We must have $|a| = 1$ or 5 as these are the only common divisors of 15 and 10.

<u>Case 1</u>: $|a| = 1 \Rightarrow |b| = 10$. Then $a = \bar{0}$ and there are $\varphi(10) = 4$ choices for b in $\mathbb{Z}_{20}$, giving 4 elements.

<u>Case 2</u>: $|a| = 5$. Then either $|b| = 2$ or $|b| = 10$. If $|b| = 2$, this gives $\varphi(5) \cdot \varphi(2) = 4 \cdot 1 = 4$ elements; if $|b| = 10$, we get $\varphi(5)\varphi(10) = 4 \cdot 4 = 16$ elements.

Thus, in total there are $\boxed{24}$ elements of order 10. Since every cyclic subgroup of order 10 contains $\varphi(10) = 4$ elements of order 10, and each of these elements lies in a unique cyclic subgroup, there are $\frac{24}{4} = \boxed{6}$ cyclic subgroups of order 10 in G.

4. Let G be an abelian group, $n \in \mathbb{Z}^+$. Define $H = \{g \in G \mid g^n = e\}$ and $K = \{g^n \mid g \in G\}$. Prove that $G/H \cong K$.

Proof Define $\phi: G \to K$ by $\phi(g) = g^n$. Let $g, h \in G$; since G is abelian we have $\phi(gh) = (gh)^n = g^n h^n = \phi(g)\phi(h)$, so ϕ is a homomorphism. Let $k \in K$. Then $k = g^n$ for some $g \in G$ and $k = \phi(g)$, so ϕ is surjective. Note that $\mathrm{Ker}(\phi) = \{g \in G \mid g^n = e\} = H$. Therefore, by the First Isomorphism Theorem, we have $G/\mathrm{Ker}(\phi) \cong \phi(G) \Rightarrow G/H \cong K$. ▨

5. (a) How many homomorphisms are there $\phi: \mathbb{Z}_{45} \to \mathbb{Z}_{30}$?

A homom. is completely determined by $\phi(\bar{1})$, which must divide 30 by Lagrange's Thrm, and divides $|\bar{1}| = 45$ by properties of homom. Thus, $|\phi(\bar{1})|$ divides $\gcd(30, 45) = 15$. So $|\phi(\bar{1})| \in \{1, 3, 5, 15\}$. Counting the # of elements of these orders, we have $\varphi(1) + \varphi(3) + \varphi(5) + \varphi(15) = 1 + 2 + 4 + 8 = \boxed{15}$ homom.

(b) Suppose that $\phi(\bar{1}) = \bar{2}$. Find $\phi^{-1}(\bar{8})$.

Now $\mathrm{Im}(\phi) = \langle \bar{2} \rangle \Rightarrow |\mathrm{Im}(\phi)| = 15$, so by FIT, $|\mathrm{Ker}(\phi)| = \dfrac{|G|}{|\mathrm{Im}(\phi)|} = \dfrac{45}{15} = 3$.

The unique subgp of $\mathbb{Z}_{45}$ of order 3 is $\langle \overline{15} \rangle = \{\bar{0}, \overline{15}, \overline{30}\}$.

Since $\phi(\bar{4}) = 4\phi(\bar{1}) = 4 \cdot \bar{2} = \bar{8}$, we have that

$$\phi^{-1}(\bar{8}) = \bar{4} + \mathrm{Ker}(\phi) = \boxed{\{\bar{4}, \overline{19}, \overline{34}\}}.$$

6. Classify all abelian groups of order 540.

Note that $540 = 54 \cdot 10 = 2 \cdot 27 \cdot 2 \cdot 5 = 2^2 \cdot 3^3 \cdot 5$. So there are $2 \cdot 3 \cdot 1 = 6$ isomorphism classes:

$\mathbb{Z}_4 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_5$ $\quad$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_5$

$\mathbb{Z}_4 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ $\quad$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$

$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ $\quad$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$.

If G is an abelian gp of order 540 with no element of order 27, this eliminates the top row. Having an element of order 9 eliminates the bottom row. Since G has three elements of order 2, it cannot have the factor $\mathbb{Z}_4$ which has only 1 element of order 2, so

$$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5.$$