$$\hat{A} = \begin{bmatrix} 1 & 3 \\ 12 & 1 \end{bmatrix} \qquad p(r) = \begin{vmatrix} 1-r & 3 \\ 12 & 1-r \end{vmatrix} = r^2 - 2r - 35 = 0$$

$$(r-7)(r+5) = 0$$

Eigenvalues $r = 7$, $r = -5$

$$r = 7: \begin{bmatrix} -6 & 3 \\ 12 & -6 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \hat{0} \Rightarrow \begin{matrix} 3u_2 = 6u_1 \\ u_2 = 2u_1 \end{matrix} \qquad \hat{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$r = -5: \begin{bmatrix} 6 & 3 \\ 12 & 6 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \hat{0} \Rightarrow \begin{matrix} 3u_2 = -6u_1 \\ u_2 = -2u_1 \end{matrix} \qquad \hat{u}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Gen. soln $\hat{x}(t) = c_1 e^{7t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{vmatrix} e^t & \sin t & -\cos t \\ e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \end{vmatrix} = e^t(\cos^2 t + \sin^2 t) - \sin t\,(e^t \cos t - e^t \sin t) - \cos t\,(-e^t \sin t - e^t \cos t)$$

$$= e^t - e^t \cancel{\sin t \cos t} + e^t \sin^2 t + e^t \cancel{\sin t \cos t} + e^t \cos^2 t = 2e^t \neq 0$$

7.2#2

1) $\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2\, dt$

$$= \lim_{N\to\infty} -\left[\frac{t^2 e^{-st}}{s} + \frac{2te^{-st}}{s^2} + \frac{2e^{-st}}{s^3}\right]_0^N$$

$$= \lim_{N\to\infty}\left[\frac{2}{s^3} - \frac{N^2 e^{-sN}}{s} - \frac{2Ne^{-sN}}{s^2} - \frac{2e^{-sN}}{s^3}\right]$$

$$= \frac{2}{s^3}, \quad s > 0$$

$$\begin{array}{ccc} t^2 & + & e^{-st} \\ 2t & - & -\frac{e^{-st}}{s} \\ 2 & + & \frac{e^{-st}}{s^2} \\ 0 & & -\frac{e^{-st}}{s^3} \end{array}$$

7.2 #4 $\mathcal{L}\{te^{3t}\} = \int_0^\infty e^{-st}te^{3t}dt$

$$= \int_0^\infty te^{-(s-3)t}dt$$

$$= \lim_{N\to\infty}\left[-\frac{te^{-(s-3)t}}{s-3} - \frac{e^{-(s-3)t}}{(s-3)^2}\right]_0^N$$

$$= \lim_{N\to\infty}\left[\frac{-Ne^{-(s-3)N}}{s-3} - \frac{e^{-(s-3)N}}{(s-3)^2} + \frac{1}{(s-3)^2}\right] = \frac{1}{(s-3)^2} \text{ for } s > 3$$

IBP

$$\begin{array}{ccc} t & + & e^{-(s-3)t} \\ 1 & - & -\frac{e^{-(s-3)t}}{s-3} \\ 0 & & \frac{e^{-(s-3)t}}{(s-3)^2} \end{array}$$

RP7#2

$$\mathcal{L}\{f(t)\} = \int_0^5 e^{-st}e^{-t}\,dt + \int_5^\infty -e^{-st}\,dt$$

$$= \int_0^5 e^{-(s+1)t}\,dt + \int_5^\infty -e^{-st}\,dt$$

$$= -\frac{e^{-(s+1)t}}{s+1}\Big|_0^5 + \lim_{N\to\infty}\frac{e^{-st}}{s}\Big|_5^N = \boxed{\frac{1}{s+1} - \frac{e^{-5(s+1)}}{s+1} - \frac{e^{-5s}}{s}} \text{ for } s > 0$$

(2) $\boxed{13}$ $\dfrac{4s^2 + 13s + 19}{(s-1)(s^2+4s+13)} = \dfrac{A}{s-1} + \dfrac{B(s+2)+3C}{(s+2)^2+3^2} \Rightarrow A(s^2+4s+13) + \boxed{B(s+2)+3C}(s-1) = 4s^2 + 13s + 19$

Take $s = 1$: $18A = 36 \Rightarrow A = 2$

$s = -2$: $9A - 9C = 9 \Rightarrow -9C = -9 \Rightarrow C = 1$

$s = 0$: $13A - 2B - 3C = 19 \Rightarrow -2B = -4 \Rightarrow B = 2$

$$\mathcal{L}^{-1}\left\{\frac{2}{s-1} + \frac{2(s+2) + 3}{(s+2)^2 + 3^2}\right\} = 2e^t + 2e^{-2t}\cos(3t) + e^{-2t}\sin(3t)$$

$\boxed{14}$ $\dfrac{s^2+16s+9}{(s+1)(s+3)(s-2)} = \dfrac{A}{s+1} + \dfrac{B}{s+3} + \dfrac{C}{s-2} \Rightarrow A(s+3)(s-2) + B(s+1)(s-2) + C(s+1)(s+3) = s^2 + 16s + 9$

Take $s = -1$: $-6A = -6 \Rightarrow A = 1$

$s = -3$: $10B = -30 \Rightarrow B = -3$

$s = 2$: $15C = 45 \Rightarrow C = 3$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{3}{s+3} + \frac{3}{s-2}\right\} = e^{-t} - 3e^{-3t} + 3e^{2t}$$

$\boxed{15}$ $\dfrac{2s^2 + 3s - 1}{(s+1)^2(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+2} \Rightarrow A(s+1)(s+2) + B(s+2) + C(s+1)^2 = 2s^2 + 3s - 1$

Take $s = -1$: $B = -2$

$s = -2$: $C = 1$

$s = 0$: $2A + 2B + C = 2A - 3 = -1$

$\Rightarrow 2A = 2 \Rightarrow A = 1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{1}{s+2}\right\} = e^{-t} - 2te^{-t} + e^{-2t}$$

(3) **19** $y'' - 7y' + 10y = 0 \qquad y(0) = 0, \; y'(0) = -3$

$$\mathcal{L}[y''] - 7\mathcal{L}[y'] + 10\mathcal{L}[y] = 0$$

$$(s^2 Y(s) + 3) - 7(sY(s)) + 10Y(s) = 0$$

$$(s^2 - 7s + 10)Y(s) + 3 = 0$$

$$Y(s) = \frac{-3}{s^2 - 7s + 10} = \frac{A}{(s-2)} + \frac{B}{(s-5)} \qquad A = 1, \; B = -1$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s-2} - \frac{1}{s-5} \right\} = \underline{e^{2t} - e^{5t}}$$

20 $y'' + 6y' + 9y = 0 \qquad y(0) = -3, \; y'(0) = 10$

$$\mathcal{L}[y''] + 6\mathcal{L}[y'] + 9\mathcal{L}[y] = 0$$

$$(s^2 Y(s) + 3s - 10) + 6(sY(s) + 3) + 9Y(s) = 0$$

$$(s^2 + 6s + 9)Y(s) + 3s + 8 = 0$$

$$Y(s) = \frac{-3s - 8}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2} \Rightarrow -3s - 8 = A(s+3) + B = As + (3A + B)$$

$$\Rightarrow A = -3, \quad B = -8 - 3A = -8 + 9 = 1$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{-3}{s+3} + \frac{1}{(s+3)^2} \right\} = \underline{-3e^{-3t} + te^{-3t}}$$

22 $y'' + 9y = 10e^{2t} \qquad y(0) = -1, \; y'(0) = 5$

$$\mathcal{L}[y''] + 9\mathcal{L}[y] = 10\mathcal{L}[e^{2t}]$$

$$s^2 Y(s) + s - 5 + 9Y(s) = \frac{10}{s-2}$$

$$(s^2 + 9)Y(s) = \frac{10 - (s-5)(s-2)}{s-2} = \frac{\cancel{10} - s^2 + 7s \cancel{-10}}{s-2}$$

$$Y(s) = \frac{-s^2 + 7s}{(s-2)(s^2+9)} = \frac{A}{s-2} + \frac{Bs + 3C}{s^2 + 3^2} \Rightarrow A(s^2+9) + [Bs + 3C](s-2) = -s^2 + 7s$$

$s = 2$: $13A = 10 \Rightarrow \underline{A = \frac{10}{13}}$

$s = 0$: $9A - 6C = 0 \Rightarrow 6C = \frac{90}{13} \Rightarrow \underline{C = \frac{15}{13}}$

$s = 1$: $10A - B - 3C = 6$

$\frac{55}{13} - B = 6 \Rightarrow B = \frac{55 - 78}{13} = \underline{\frac{-23}{13}}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{10/13}{s-2} + \frac{-23/13\, s + 3(15/13)}{s^2 + 3^2} \right\}$$

$$= \underline{\frac{10}{13}e^{2t} - \frac{23}{13}\cos(3t) + \frac{15}{13}\sin(3t)}$$

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(5) 9.6 #5, #6 HW

$$A = \begin{bmatrix} -2 & -2 \\ 4 & 2 \end{bmatrix} \quad p(r) = \begin{vmatrix} -2-r & -2 \\ 4 & 2-r \end{vmatrix} = r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

$$r = 2i: \begin{bmatrix} -2-2i & -2 \\ 4 & 2-2i \end{bmatrix} \hat{z} = \hat{0} \Rightarrow z_2 = (-1-i)z_1 \Rightarrow \hat{z} = \begin{bmatrix} 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\hat{x}_1 = \cos(2t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin(2t)\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\hat{x}_2 = \sin(2t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \cos(2t)\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \boxed{\hat{X} = \begin{bmatrix} \cos(2t) & \sin(2t) \\ \sin(2t) - \cos(2t) & -\sin(2t) - \cos(2t) \end{bmatrix}}$$

RP9 #2

$$A = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \quad p(r) = \begin{vmatrix} 3-r & 2 \\ -5 & 1-r \end{vmatrix} = r^2 - 4r + 13 = 0 \quad r = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$r = 2+3i: \begin{bmatrix} 1-3i & 2 \\ -5 & -1-3i \end{bmatrix} \hat{z} = \hat{0} \Rightarrow z_2 = \left(-\tfrac{1}{2} + \tfrac{3}{2}i\right) z_1$$

$$\Rightarrow \hat{z} = \begin{bmatrix} 2 \\ -1+3i \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\boxed{\hat{X}(t) = c_1 e^{2t}\begin{bmatrix} 2\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 2\sin(3t) \\ 3\cos(3t) - \sin(3t) \end{bmatrix}}$$

(4) 9.5 #14

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix} \qquad p(r) = \begin{vmatrix} -1-r & 1 & 0 \\ 1 & 2-r & 1 \\ 0 & 3 & -1-r \end{vmatrix} = (-1-r)(r^2-r-5) + (1+r)$$

$$= -(r+1)(r^2-r-5-1)$$

$$= -(r+1)(r^2-r-6) = -(r+1)(r-3)(r+2)$$

$$r=-1: \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 0 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow \begin{matrix} u_2 = 0 \\ u_1 = -u_3 \end{matrix} \Rightarrow \hat{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$r=3: \begin{bmatrix} -4 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -4 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow \begin{matrix} u_2 = 4u_1 \\ u_3 = \frac{3}{4}u_2 = 3u_1 \end{matrix} \Rightarrow \hat{u}_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$$r=-2: \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 3 & 1 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow \begin{matrix} -u_2 = +u_1 \\ u_3 = -3u_2 \end{matrix} \Rightarrow \hat{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\boxed{\hat{x}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + c_3 e^{-2t}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}$$

~~RP9 #1~~

$$A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} \qquad p(r) = \begin{vmatrix} 6-r & -3 \\ 2 & 1-r \end{vmatrix} = r^2 - 7r + 12 = 0$$

$$(r-4)(r-3) = 0$$

$$r=3: \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow u_1 = u_2 \Rightarrow \hat{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$r=4: \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow \begin{matrix} 2u_1 = 3u_2 \\ u_1 = \frac{3}{2}u_2 \end{matrix} \quad \hat{u}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\hat{x}(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

RP9 #11

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \qquad p(r) = \begin{vmatrix} -r & 1 \\ -2 & 3-r \end{vmatrix} = r^2 - 3r + 2 = 0 \qquad \hat{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(r-2)(r-1) = 0$$

$$r=1: \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow u_1 = u_2 \Rightarrow \hat{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad r=2: \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow u_2 = 2u_1 \Rightarrow \hat{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

G.S. $\hat{x}(t) = c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\boxed{3e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$\hat{x}(0) = \begin{bmatrix} c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 1 - c_2 \Rightarrow c_1 = 3 \\ 1 - c_2 + 2c_2 = -1 \Rightarrow c_2 = -2 \end{matrix}$$

9.5 #16

$$A = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix} \qquad p(r) = \begin{vmatrix} -7-r & 0 & 6 \\ 0 & 5-r & 0 \\ 6 & 0 & 2-r \end{vmatrix} = (-7-r)(5-r)(2-r) + \overset{36(r-5)}{6(6r-30)}$$

$$= -(r-5)(r^2+5r-14-36)$$

$$= -(r-5)(r^2+5r-50) = -(r-5)^2(r+10)$$

$$r=-10: \begin{bmatrix} 3 & 0 & 6 \\ 0 & 15 & 0 \\ 6 & 0 & 12 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow \begin{matrix} u_2 = 0 \\ u_1 = -2u_3 \end{matrix} \Rightarrow \hat{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$r=5: \begin{bmatrix} -12 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & -3 \end{bmatrix} \hat{v} = \hat{0} \Rightarrow \begin{matrix} u_3 = 2v_1 \\ \hat{u} = \begin{bmatrix} s \\ v \\ 2s \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + v\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{matrix}$$

$$\boxed{\hat{x}(t) = c_1 e^{-10t}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 e^{5t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

⑥ RPT #4

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad p(r) = \begin{vmatrix} 1-r & 1 & 0 \\ 0 & 1-r & 0 \\ 0 & 0 & 2-r \end{vmatrix} = -(r-1)^2(r-2)$$

$$r=2: \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow u_1 = u_2 = 0 \Rightarrow \hat{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$r=1: (A-I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (A-I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{u} = \hat{0}$$

$$\Rightarrow u_3 = 0, \quad \hat{u} = \begin{bmatrix} s \\ v \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\hat{x}_2 = e^t\left(I\hat{u}_1 + t(A-I)\hat{u}_1\right) = e^t\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = e^t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{x}_3 = e^t\left(I\hat{u}_2 + t(A-I)\hat{u}_2\right) = e^t\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} te^t \\ e^t \\ 0 \end{bmatrix}$$

$$\hat{X} = \begin{bmatrix} 0 & e^t & te^t \\ 0 & 0 & e^t \\ e^{2t} & 0 & 0 \end{bmatrix} \Rightarrow \hat{X}(0) = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \hat{X}(0)^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$e^{\hat{A}t} = \hat{X}(t)\hat{X}(0)^{-1} = \begin{bmatrix} 0 & e^t & te^t \\ 0 & 0 & e^t \\ e^{2t} & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}}$$

7.8 #8

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \qquad p(r) = \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = r^2 - 2r - 3 = 0 \Rightarrow (r-3)(r+1) = 0$$

$$r=-1: \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow u_2 = -2u_1 \Rightarrow \hat{u}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \hat{X} = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix}$$

$$r=3: \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \hat{u} = \hat{0} \Rightarrow u_2 = 2u_1 \Rightarrow \hat{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\hat{X}(0) = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \Rightarrow \hat{X}(0)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow e^{\hat{A}t} = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \boxed{\frac{1}{4}\begin{bmatrix} 2e^{-t} + 2e^{3t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{-t} + 2e^{3t} \end{bmatrix}}$$

6) RP9 #16 ✓

$A = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad p(r) = \begin{vmatrix} -r & 1 & 4 \\ 0 & -r & 2 \\ 0 & 0 & -r \end{vmatrix} = -r^3$

$r=0$: $A^2 = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \hat{0} \Rightarrow u_1, u_2, u_3$ all free

$\Rightarrow \hat{u} \begin{bmatrix} s \\ v \\ w \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ see below

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9.8 #10 X

$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$ $\quad p(r) = \begin{vmatrix} -r & 2 & 2 \\ 2 & -r & 2 \\ 2 & 2 & -r \end{vmatrix} = -r(r^2-4) - 2(-2r-4) + 2(4-2r) =$ $+4(r+2)$ $-4(r-2)$

RP9 #6 X

$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 3 & 4 \end{bmatrix}$ $\quad p(r) = \begin{vmatrix} 5-r & 0 & 0 \\ 0 & -4-r & 3 \\ 0 & 3 & 4-r \end{vmatrix} = (5-r)(r^2-25) = -(r-5)^2(r+5)$

$r=-5$: $\begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix}\hat{u} = \hat{0} \Rightarrow \begin{matrix} u_1 = 0 \\ u_2 = -3u_3 \end{matrix} \Rightarrow \hat{u}_1 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$

$r=5$: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -9 & 3 \\ 0 & 3 & -1 \end{bmatrix}\hat{u} = \hat{0} \Rightarrow u_3 = 3u_2 \Rightarrow \hat{u} = \begin{bmatrix} s \\ v \\ 3v \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$\hat{X} = \begin{bmatrix} 0 & e^{5t} & 0 \\ -3e^{-5t} & 0 & e^{5t} \\ e^{-5t} & 0 & 3e^{5t} \end{bmatrix} \Rightarrow \hat{X}(0) = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{10} & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{3}{10} & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{10} & 0 \end{array}\right]$ $\quad e^{\hat{A}t} = \begin{bmatrix} 0 & e^{5t} & 0 \\ -3e^{-5t} & 0 & e^{5t} \\ e^{-5t} & 0 & 3e^{5t} \end{bmatrix}\begin{bmatrix} 0 & -3/10 & 1 \\ 1 & 0 & 0 \\ 0 & 1/10 & 0 \end{bmatrix}$

$= \begin{bmatrix} e^{5t} & 0 & 0 \\ 0 & \frac{9}{10}e^{-5t} + \frac{1}{10}e^{5t} & -3e^{-5t} \\ 0 & -\frac{3}{10}e^{-5t} + \frac{3}{10}e^{5t} & e^{-5t} \end{bmatrix}$

16, cont

$\Rightarrow \hat{X}_1 = \hat{I}\hat{u}_1 + t\hat{A}\hat{u}_1 + \frac{t^2}{2}(\hat{A}^2)\hat{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \hat{X}_2 = I\hat{u}_2 + tA\hat{u}_2 + \frac{t^2}{2}A^2\hat{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\hat{X}_3 = I\hat{u}_3 + tA\hat{u}_3 + \frac{t^2}{2}A^2\hat{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$\hat{X} = \boxed{\begin{bmatrix} 1 & t & 4t+t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \hat{X}(0) = I = \hat{X}(0)^{-1} \Rightarrow \hat{X} = e^{\hat{A}t}$