LINE AND SURFACE INTEGRALS: A SUMMARY OF CALCULUS 3 UNIT 4

The final unit of material in multivariable calculus introduces many unfamiliar and non-intuitive concepts in a short amount of time. This document attempts to succinctly describe this material and enable the student to draw connections and links between concepts. There are two main issues with this material: first, understanding how to compute the various integrals introduced and what they represent; second, knowing what shortcuts can be used in these computations and the proper situation in which to employ their effectiveness.

1. Parameterizations and the Computation and Interpretation of Line and Surface Integrals

There are two basic types of integrals we compute: line integrals and surface integrals. This distinction describes the object over which we are integrating, i.e. the domain of integration. In the case of line integrals, we integrate over some curve in $\mathbb{R}^2$ or $\mathbb{R}^3$; the most popular curves are line segments and circles, but they can be more complicated. One will either be given or must write for oneself a parametrization, which will depend on a single variable. In the case of surface integrals, we integrate over a surface in $\mathbb{R}^3$; popular surfaces include cylinders, spheres, and planes, but again could be more complicated. Once more, a parametrization must be provided to you or derived, which will depend on two variables. Corresponding to the parametrization of a surface is a normal vector, which is orthogonal to the surface and whose orientation is consistent with directions in the problem or a standard convention (i.e. outward from a closed surface). Below we give the standard parameterizations for the most common curves and surfaces. (Note that if the problem restricts these objects, such as taking only the upper half of a circle or sphere, bounds for the parameters must be adjusted accordingly.)

Example 1. To parameterize a line segment from (x_1, y_1, z_1) to (x_2, y_2, z_2), we write

$$\mathbf{r}(t) = \langle (x_2 - x_1)t + x_1, (y_2 - y_1)t + y_1, (z_2 - z_1)t + z_1 \rangle, 0 \le t \le 1.$$

If two of the three coordinates, say y and z, remain fixed, an easy and simple alternate parametrization is

$$\mathbf{r}(t) = \langle t, y, z \rangle, x_1 \le t \le x_2.$$

Example 2. For a circle of radius R centered at the origin, $\mathbf{r}(t) = \langle R\cos t, R\sin t \rangle$ for $0 \le t \le 2\pi$.

Example 3. For a cylinder of radius R with central axis the z-axis,

$$\mathbf{r}(\theta, z) = \langle R\cos\theta, R\sin\theta, z \rangle, 0 \le \theta \le 2\pi, -\infty < z < \infty.$$

Often the z value will be restricted over a certain interval. If the central axis is not the z-axis, a similar parametrization works where the central axis is left free and the other two variables operate with polar-type coordinates. The normal vector in this case is $\mathbf{n} = \langle R\cos\theta, R\sin\theta, 0\rangle, ||\mathbf{n}|| = R$.

Example 4. For a plane $ax+by+cz=d$, we write $\mathbf{r}(x,y) = \langle x, y, (d-ax-by)/c\rangle$; most often we only consider the portion in the first octant (where all variables are positive), so the bounds for parameters x and y can be found by looking at the projection into the x,y-plane $ax+by=d$. Whenever z is written as a function of x and y like it is here, we have $\mathbf{n} = \langle -k_x, -k_y, 1\rangle, ||\mathbf{n}|| = \sqrt{1+k_x^2+k_y^2}$.

Example 5. For a sphere of radius R centered at the origin,

$$\mathbf{r}(\theta,\phi) = \langle R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi\rangle, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi.$$

It can be shown that in this case $\mathbf{n} = R^2\sin\phi\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$ and $||\mathbf{n}|| = R^2\sin\phi$.

Now that we are able to write parameterizations, let us observe a second distinction between these integrals: the object we are integrating (i.e. the integrand). For both line and surface integrals, we may integrate either a scalar-valued function $f(x,y,z)$ or a vector field $\mathbf{F} = \langle f, g, h\rangle$. For each of these possibilities, the formulas for computing the line and surface integral are very similar.

Definition 1. The line integral of $f(x,y)$ over a curve $\mathcal{C}$ parameterized by $\mathbf{r}(t)$ is calculated as follows:

$$\int_{\mathcal{C}} f\,ds = \int_{\alpha}^{\beta} f(\mathbf{r}(t))||\mathbf{r}'(t)||\,dt.$$

Definition 2. The surface integral of $f(x,y,z)$ over the surface $\mathcal{S}$ parameterized by $\mathbf{r}(u,v)$ with domain D is calculated as follows:

$$\iint_{\mathcal{S}} f\,dS = \iint_{D} f(\mathbf{r}(u,v))||\mathbf{n}||\,dudv,$$

where $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$.

It is when integrating over a function that we need to find a magnitude. Note that the main difference in these formulas, other than the parametrization, is that $\mathbf{r}'(t)$ is replaced by the normal vector $\mathbf{n}$. The same holds for the integrals over a vector field.

Definition 3. The line integral of $\mathbf{F} = \langle f, g, h\rangle$ over a curve $\mathcal{C}$ parameterized by $\mathbf{r}(t)$ is calculated by

$$\int_{\mathcal{C}} \mathbf{F}\,d\mathbf{r} = \int_{\alpha}^{\beta} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt.$$

Definition 4. The surface integral of $\mathbf{F}$ over the surface $\mathcal{S}$ parameterized by $\mathbf{r}(u,v)$ with domain D is calculated by

$$\iint_{\mathcal{S}} \mathbf{F}\,d\mathbf{S} = \iint_{D} \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{n}\,dudv,$$

where $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$.

The main difference when integrating over a vector field is that we no longer have to find a magnitude, but rather a dot product.

Finally, many students ask "what am I actually calculating when I find these integrals?", so let us briefly comment on their interpretation. For the line integral of a function $f(x, y)$, observe that f takes on various z-values as x and y range over the points on the curve. This creates a sort of "wall" or "sheet" if we drop perpendiculars from the z-values down to the x, y-plane (or up to the x, y-plane if the z-value is negative). The line integral calculates the (signed) area of this sheet. The surface integral of a function is similar, except that the function must take three inputs $f(x, y, z)$, so the output lives in $\mathbb{R}^4$ and our visual fails us. However, we do know that if we take the function to be identically 1, we get the surface area of the surface over which we are integrating.

In the case of the line integral of a vector field, we are finding how much the vector field points in the direction of the curve $\mathcal{C}$. If the vector field is a force field, the integral gives the work done by the field in moving an object along the curve. For the surface integral of a vector field, we are finding how much the vector field points in a direction parallel to the normal vector of the surface, that is, how much the vector field passes through the surface; this is also called the flux.

2. Shortcuts and When to Use Them

The first shortcut applies to line integrals of vector fields. The computation is simpler when the vector field is conservative, as the following theorem says.

Theorem 6. (Fundamental Theorem of Conservative Vector Fields) Let $\mathbf{F}$ be a continuous, conservative vector field with $\mathbf{F} = \nabla\phi$ in an open connected region D (in $\mathbb{R}^2$ or $\mathbb{R}^3$). If $\mathcal{C}$ is a smooth oriented curve from P to Q in the region D, then $\displaystyle\int_{\mathcal{C}} \mathbf{F}\, d\mathbf{r} = \phi(Q) - \phi(P)$.

So when faced with computing such a line integral, it is wise to first check whether the vector field $\mathbf{F} = \langle F_1, F_2, F_3\rangle$ is conservative. For most vector fields (those with simply connected, open domains) this amounts to satisfying the following three cross-partial conditions:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$

Having done this, how does one find the potential function ϕ such that $\nabla\phi = \mathbf{F}$? My method is to integrate each component with respect to the corresponding variable, and select each term that appears, not counting repeats. This technique is illustrated in the following example.

Example 7. Suppose we want to find the potential function for the vector field $\mathbf{F} = \langle y + z, x + z, x + y\rangle$. We perform three partial integrations:

$$\int (y + z)dx = xy + xz + f(y, z)$$

$$\int (x + z)dy = xy + yz + g(x, z)$$

$$\int (x+y)dz = xz + yz + h(x,y)$$

Therefore, the terms that appear (not counting repetition) are $\phi(x,y,z) = xy + xz + yz$.

The remaining shortcuts should be applied with more discernment. The next also applies to line integrals of vector fields.

Theorem 8. (Green's Theorem) Let D be a simply connected region of $\mathbb{R}^2$ whose boundary curve $\mathcal{C}$ is simple, closed, smooth (or piecewise smooth), and oriented counterclockwise. Let $\mathbf{F} = \langle f, g \rangle$ be continuously differentiable in D. Then

$$\oint_{\mathcal{C}} \mathbf{F}\, d\mathbf{r} = \iint_D (g_x - f_y)\, dA.$$

Essentially, Green's Theorem says that the circulation of a vector field can be evaluated by a double integral. This is especially useful when the boundary curve has multiple pieces which would require multiple parameterizations if calculated as a straightforward circulation. **When should I use Green's Theorem?** Look for line integrals of a two-dimensional vector field over a closed curve, where the region bounded by the curve can be easily represented as a double integral (rectangle, triangle, circle). Don't sweat the CCW orientation too much: if the orientation is clockwise, just negate your answer by reversing the orientation of the curve.

The final two shortcuts are related to surface integrals.

Theorem 9. (Stokes' Theorem) Let $\mathcal{S}$ be a smooth, oriented surface in $\mathbb{R}^3$ with normal vector $\mathbf{n}$ and smooth, closed boundary curve $\partial\mathcal{S} = \mathcal{C}$ whose orientation is consistent with $\mathcal{S}$. Let $\mathbf{F} = \langle f, g, h \rangle$ be continuously differentiable on $\mathcal{S}$. Then

$$\oint_{\mathcal{C}} \mathbf{F}\, d\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F})\, d\mathbf{S}.$$

Stokes' Theorem relates the circulation of a vector field around the boundary of a surface to the surface integral of that surface over a related, but <u>different</u> vector field, namely the curl of the original vector field. The difficulty with knowing when to apply Stokes' Theorem is that it is not clear which side will be easier to evaluate, as this depends on the given situation. However, since in general it is easier to find the curl of a vector field than to find the "anti-curl", as it were, we generally apply the Theorem to circulations to convert them to surface integrals. **When should I use Stokes' Theorem?** Look for circulations of a vector field where the curve is the boundary of a surface with a well-known parametrization (plane, sphere, or any function that can be explicitly solved for z).

Finally, we have the incredibly useful Divergence Theorem.

Theorem 10. (Divergence Theorem) Let $\mathbf{F}$ be a continuously differentiable vector field in a simply connected region D in $\mathbb{R}^3$ which is enclosed by an oriented surface $\mathcal{S}$. Then

$$\iint_{\mathcal{S}} \mathbf{F}\, d\mathbf{S} = \iiint_D \operatorname{div}(\mathbf{F})\, dV.$$

This theorem simplifies the surface integral of a vector field into a triple integral of a related scalar-valued function, namely div($\mathbf{F}$). As with Green's Theorem, this is especially useful when the surface consists of many parts which would require multiple parameterizations if done in the usual way. **When should I use the Divergence Theorem?** Look for surface integrals of vector fields where the interior of the surface can be easily represented as a triple integral (box, sphere, cylinder).