MAP 2302

Section 4787

Final Exam

August 5, 2016

Name: Solutions

This exam consists of 7 free-response problems. There are 52 points possible, but the exam is only counted out of 50 points (so it is possible to get 104%).

You may not use any books, notes, or calculators on this exam.

You are required to show sufficient work for each problem on the exam. For the sake of your instructor (grader), please abide by the following guidelines:

- Organize your work in a neat and coherent way in the space provided. Please try not to scatter work all over the page in a complicated fashion. You may want to use scratch paper when first solving the problem, and then neatly copy your relevant and organized work onto the test.
- Answers which are not justified will not receive full credit. You must show sufficient work to indicate you understand all the steps involved in solving the problem. A correct answer with no supporting work will receive little, if any, credit, but an incorrect final answer with accompanying work will receive credit proportional to the accuracy of the work.
- If you need more space for a problem, use a (clean) piece of scratch paper and clearly indicate which problem you are solving on the page, as well as notifying your instructor of this when you turn in the exam.

You will have 75 minutes to complete the exam.

Your signature below indicates that you promise to abide by the UF Honor Code.

I have neither given nor received unauthorized help on this exam.

Signature ______________________________

Do your best, and good luck!

1. (6 points) Use the <u>definition</u> of the Laplace transform to find $\mathscr{L}\{te^{6t}\}$, and state the domain of this function.

$$\mathcal{L}\{te^{6t}\} = \int_0^\infty e^{-st}\, te^{6t}\,dt = \lim_{N\to\infty} \int_0^N te^{-(s-6)t}\,dt$$

$$= \lim_{N\to\infty}\left[-\frac{te^{-(s-6)t}}{s-6} - \frac{e^{-(s-6)t}}{(s-6)^2}\right]_0^N$$

IBP

$$\begin{array}{ccc} & & e^{-(s-6)t} \\ t & + & -\frac{e^{-(s-6)t}}{s-6} \\ 1 & - & \frac{e^{-(s-6)t}}{(s-6)^2} \\ 0 & & \end{array}$$

$$= \lim_{N\to\infty}\left[\frac{1}{(s-6)^2} - \frac{Ne^{-(s-6)N}}{s-6} - \frac{e^{-(s-6)N}}{(s-6)^2}\right]$$

$$= \boxed{\frac{1}{(s-6)^2}}$$

Domain: $\boxed{s > 6}$

2. (9 points) Find the inverse Laplace transform of the function

$$F(s) = \frac{3s^2 - 10s + 21}{(s+1)(s^2 - 8s + 25)}.$$

$$\frac{3s^2 - 10s + 21}{(s+1)[(s-4)^2 + 3^2]} = \frac{A}{s+1} + \frac{B(s-4) + 3C}{(s-4)^2 + 3^2}$$

$$\Rightarrow 3s^2 - 10s + 21 = A(s^2 - 8s + 25) + [B(s-4) + 3C](s+1)$$

Take $s = -1$: $34A = 34 \Rightarrow \underline{A = 1}$

Take $s = 4$: $9A + 15C = 29 \Rightarrow 15C = 29 - 9 = 20 \Rightarrow \underline{C = \frac{4}{3}}$

Take $s = 0$: $25A - 4B + 3C = 29 - 4B = 21 \Rightarrow -4B = -8 \Rightarrow \underline{B = 2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s-4}{(s-4)^2+3^2}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s-4)^2+3^2}\right\}$$

$$= \boxed{e^{-t} + 2e^{4t}\cos(3t) + \frac{4}{3}e^{4t}\sin(3t)}$$

3. (11 points) Use Laplace transforms to solve the initial value problem

$$y'' - 4y' + 4y = 2e^t, \quad y(0) = 0, \quad y'(0) = 0.$$

$$\mathcal{L}[y''] - 4\mathcal{L}[y'] + 4\mathcal{L}[y] = 2\mathcal{L}[e^t]$$

$$\mathcal{L}[y''] = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s)$$

$$\mathcal{L}[y'] = sY(s) - y(0) = sY(s)$$

$$s^2 Y(s) - 4sY(s) + 4Y(s) = \frac{2}{s-1}$$

$$Y(s)(s^2 - 4s + 4) = \frac{2}{s-1}$$

$$Y(s) = \frac{2}{(s-1)(s-2)^2}$$

PFD

$$\frac{2}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$\Rightarrow 2 = A(s-2)^2 + B(s-1)(s-2) + C(s-1)$$

Take $s = 1$: $A = 2$

Take $s = 2$: $C = 2$

Take $s = 0$: $4A + 2B - C = 2$

$8 + 2B - 2 = 2$

$2B = -4$

$B = -2$

$$\Rightarrow Y(s) = \frac{2}{s-1} - \frac{2}{s-2} + \frac{2}{(s-2)^2}$$

$$\Rightarrow y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y(t) = 2e^t - 2e^{2t} + 2te^{2t}}$$

4. (6 points) Find a general solution to the system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t),$$

given that the characteristic polynomial is $p(r) = -(r-2)(r+1)^2$.

Eigenvalues $r = 2$, $r = -1$ (mult. 2)

$r = 2$: $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \hat{0} \Rightarrow$ $2u_1 = u_2 + u_3 = 2u_3 \Rightarrow u_1 = u_3$

$-3u_2 + 3u_3 = 0$

$\Rightarrow u_2 = u_3$

so $\hat{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$r = -1$: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \hat{0} \Rightarrow$ $u_1 = -u_2 - u_3$

Let $u_2 = s$, $u_3 = v$. Then

$\hat{u} = \begin{bmatrix} -s-v \\ s \\ v \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad \hat{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\hat{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

General solution: $\boxed{\hat{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$.

5. (5 points) Find two linearly independent real-valued vector solutions to the system

$$\mathbf{x}'(t) = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix} \mathbf{x}(t).$$

Char. polyn. $p(r) = \begin{vmatrix} -2-r & -5 \\ 1 & 2-r \end{vmatrix} = r^2 + 1 = 0 \Rightarrow r^2 = -1$

$\Rightarrow \underline{r = \pm i}$

$r = i: \begin{bmatrix} -2-i & -5 \\ 1 & 2-i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \hat{0} \Rightarrow z_1 = (-2+i) z_2$

$\Rightarrow \hat{z} = \begin{bmatrix} -2+i \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\hat{X}_1(t) = e^{0t} \cdot \left(\cos t \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \boxed{\begin{bmatrix} -2\cos t - \sin t \\ \cos t \end{bmatrix}}$

$\hat{X}_2(t) = e^{0t} \left(\sin t \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \boxed{\begin{bmatrix} -2\sin t + \cos t \\ \sin t \end{bmatrix}}$

6. (4 points) Decide whether the following statements are true or false.

(a) Every continuous function has a Laplace transform.

False [Counterexample: $f(t) = e^{t^2}$]

(b) If $Y(s) = \mathscr{L}\{y(t)\}$, then $\mathscr{L}\{ty'(t)\} = sY'(s) + Y(s)$.

False $\left[\mathscr{L}\{ty'(t)\} = -\frac{d}{ds}\mathscr{L}\{y'\} = -\frac{d}{ds}(sY(s) - y(0)) = -sY'(s) - Y(s) \right]$

(c) If $\mathbf{A}$ has distinct eigenvalues $r_1, \ldots, r_n$ and $\mathbf{u}_i$ is an eigenvector associated with r_i, then $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent.

True [Section 9.5, Theorem 2]

(d) If $\mathbf{A}$ is any matrix, then $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \ldots + \mathbf{A}^n\frac{t^n}{n!}$.

False [infinite series, although statement is true for nilpotent matrices]

7. Consider the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The characteristic polynomial is $p(r) = -r(r-1)^2$, and an eigenvector for the eigenvalue $r = 0$ is $\mathbf{u}_1 = [\ 1 \quad 0 \quad 0\ ]^T$.

(a) (7 points) Find a fundamental matrix $\mathbf{X}(t)$ for the system.

Find generalized eigenvectors for $r=1$:

$$\hat{A}-\hat{I} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\hat{A}-\hat{I})^2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(\hat{A}-\hat{I})^2\hat{u} = \hat{0} \Rightarrow u_1 - u_3 = 0 \Rightarrow u_1 = u_3 \Rightarrow \hat{u} = \begin{bmatrix} v \\ s \\ v \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $u_2 = s,\ u_3 = v$

$$\hat{x}_1(t) = e^t(\hat{I}\hat{u}_1 + t(\hat{A}-\hat{I})\hat{u}_1) = e^t\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix}$$

$$\hat{x}_2(t) = e^t(\hat{I}\hat{u}_2 + t(\hat{A}-\hat{I})\hat{u}_2) = e^t\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} e^t \\ 2te^t \\ e^t \end{bmatrix}$$

Fundamental matrix: $\boxed{\hat{X} = \begin{bmatrix} 1 & 0 & e^t \\ 0 & e^t & 2te^t \\ 0 & 0 & e^t \end{bmatrix}}$

(b) (2 points) Find the inverse matrix $\mathbf{X}(0)^{-1}$.

$$\hat{X}(0) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

R1−R3: $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \hat{X}(0)^{-1}$$

(c) (2 points) Find the matrix exponential $e^{\mathbf{A}t}$.

$$e^{\hat{A}t} = \hat{X}(t)\hat{X}(0)^{-1} = \begin{bmatrix} 1 & 0 & e^t \\ 0 & e^t & 2te^t \\ 0 & 0 & e^t \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & e^t-1 \\ 0 & e^t & 2te^t \\ 0 & 0 & e^t \end{bmatrix}}$$