Chapter 7

3. Let $H = \{0, \pm 3, \pm 6, \pm 9, \ldots \}$. Find all the left cosets of $H$ in $\mathbb{Z}$.

- $H + H = 3 + H = 6 + H$, etc.
- $1 + H = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots \} = 4 + H = 7 + H$, etc.
- $2 + H = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots \} = 5 + H = 8 + H$, etc.

5. Let $H$ be as in #3. Decide whether the following cosets of $H$ are the same.

We know that $a + H = b + H$ iff $b - a \in H$.

(a) $11 + H$ and $17 + H$: since $17 - 11 = 6 \in H$, these cosets are the same.
(b) $-1 + H$ and $5 + H$: since $5 - (-1) = 6 \in H$, these cosets are the same.
(c) $7 + H$ and $23 + H$: since $23 - 7 = 16 \notin H$, these cosets are not the same.

7. Find all of the left cosets of $\{7, 11, 15\}$ in $U(30)$.

Recall that $U(30) = \{1, 7, 11, 13, 17, 19, 23, 29\}$. Let $H = \{7, 11, 15\}$.

We have $7H = 11H = 15H = H$,

$7H = \{7, 17, 27\} = \{7, 11, 15\} = 7H$,

$13H = \{13, 19, 25\} = \{13, 19, 25\} = 13H$,

$19H = \{19, 29\} = \{19, 29\} = 19H$.

8. Suppose that $|a| = 15$. Find all of the left cosets of $\langle a^5 \rangle$ in $\langle a \rangle$.

Let $H = \langle a^5 \rangle = \{e, a^5, a^{10}\}$. The left cosets of $H$ are

- $eH = a^5H = a^{10}H = H$,
- $aH = \{a, a^6, a^{11}\} = a^6H = a^{11}H$,
- $a^2H = \{a^2, a^7, a^{12}\} = a^7H = a^{12}H$,
- $a^3H = \{a^3, a^8, a^{13}\} = a^8H = a^{13}H$,
- $a^4H = \{a^4, a^9, a^{14}\} = a^9H = a^{14}H$.

9. Let $|a| = 30$. How many left cosets of $\langle a^4 \rangle$ in $\langle a \rangle$ are there? List them.

Let $H = \langle a^4 \rangle = \{e, a^4, a^8, a^{12}, a^{16}, a^{20}, a^{24}, a^2, a^6, a^{10}, a^{14}, a^{18}, a^{22}, a^{26}\}$.

There are $2$ left cosets of $H$ in $\langle a \rangle$: one is $H$ and the other is

$aH = \{a, a^7, a^{13}, a^{19}, a^5, a^{11}, a^6, a^{12}, a^2, a^8\}$.

15. Let $G$ be a group of order 60. What are the possible orders for the subgroups of $G$?

By Lagrange's Theorem, the possible orders for a subgroup are the divisors of the order of the group. For a group of order 60, the divisors are $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 40\}$.  

16. Suppose $K \trianglelefteq H$ and $H \trianglelefteq G$. If $|K| = 42$ and $|G| = 420$, what are the possible orders of $H$?

The possible orders for any subgroup of $G$ are $\{1, 2, 3, 4, 5, 6, 7, 10, 14, 15, 20, 21, 28, 30, 42, 60, 70, 84, 105, 140, 210, 420\}$. Since $|H| < |G|$ and $|H| > 42$, the list narrows to $\{60, 70, 84, 105, 140, 210\}$. Finally, since $K \trianglelefteq H$ we must also have $42 | |H|$ by Lagrange's Thm, so $|H| \in \{84, 210\}$.

17. Let $G$ be a group with $|G| = pq$ where $p$ and $q$ are prime. Prove that every proper subgroup of $G$ is cyclic.

**Proof.** Let $H \trianglelefteq G$ be a proper subgroup of $G$. By Lagrange's Thm, $|H| \in \{1, p, q\}$.

If $|H| = 1$, then $H = \langle e \rangle$ is cyclic; if $|H| = p$ or $|H| = q$, then $H$ is a group of prime order, hence cyclic by Corollary 5. Thus, $H$ is cyclic. □

21. Suppose $G$ is a finite group with $|G| = n$ and $me \mathbb{Z}$ with $\gcd(m, n) = 1$. If $g \in G$ and $g^m = e$, prove that $g = e$.

**Proof.** Since $g^m = e$, we know that $|g|$ divides $m$. By Corollary 3, we also know that $|g|$ divides $n$. So $|g|$ is a common divisor of $m$ and $n$, which implies $|g| \leq \gcd(m, n) = 1$. But $|g| > 1$ always holds, so in fact $|g| = 1 \Rightarrow g = e$. □


Note that $HNK$ is a subgroup of both $H$ and $K$. Therefore, by Lagrange's Thm $|HNK|$ divides both $12$ and $35$. Thus, $|HNK| = \gcd(12, 35) = 1 \Rightarrow |HNK| = 1$.

**Generalization:** If $H, K \trianglelefteq G$ and $\gcd(|H|, |K|) = 1$, then $|HNK| = 1$.

29. Let $|G| = 33$. What are the possible orders for the elements of $G$? Show that $G$ must have an element of order $3$.

**Proof.** The possible orders for the elements of $G$ are $\{1, 3, 11, 33\}$ by Corollary 3. Suppose BUCOC that $G$ has no element of order $3$. If $G$ has an element $g$ of order $33$, then $|g| = |g^3| = 33 < |G| \Rightarrow G = \langle g \rangle$ is cyclic of order $33$. Then by the Fund. Thm of Cyclic Groups, $G$ has a subgroup of order $3$ which contains $\phi(3) = 2$ elements of order $3$, a contradiction. So $G$ has no element of order $33$. Thus, $G$ consists of the identity and $33$ elements of order $11$. But each subgroup of $G$ of order $11$ contains $\phi(11) = 10$ elements of order $11$, so the total number of elements of order $11$ in $G$ must be a multiple of $10$. Since $10 \nmid 33$, this is a final contradiction. Thus, $G$ has an element of order $3$. □
31. Can a group of order 55 have exactly 20 elements of order 11? Give a reason for your answer.

No: Suppose WLOG that \(|G|=55\) and \(G\) contains exactly 20 elements of order 11. The other possible element orders are \(\{1, 5, 55\}\). If \(G\) has an element of order 55, then \(G\) is cyclic, and by the Fund. Thm of Cyclic Groups, it contains exactly 1 subgroup of order 11, which has only 10 elements of order 11, contradicting our assumption that \(G\) has 20 such elements. So \(G\) has no element of order 55, which implies there is 1 element of order 1, 20 elements of order 11, and 55 - 21 = 34 elements of order 5. But every subgroup of \(G\) of order 5 contains \(\phi(5)=4\) elements of order 5, so the number of elements of order 5 in \(G\) must be a multiple of 4.

Since 4 \(\times\) 34, this is a final contradiction. Hence, \(G\) cannot have 20 elements of order 11.

33. Let \(H, K \leq G\) where \(G\) is finite and \(H \leq K \leq G\). Prove that \(|G:H|=|G:K||K:H|\).

Proof: Since \(H \leq K\) and \(H \leq G\), we have \(H \leq K\). By Lagrange's Thm we have

\[|G:H|=\frac{|G|}{|H|} = \frac{|G|}{|K|} \cdot \frac{|K|}{|H|} = |G:K||K:H|. \]

43. Let \(G\) be a group of permutations of a set \(S\). Prove that the orbits of the members of \(S\) constitute a partition of \(S\).

Proof: First, if \(s \in S\) then \(s \in \text{orb}_G(s)\) since certainly \((i) \in G\). Thus every orbit is nonempty. This also shows that \(S = \bigcup_{s \in S} \text{orb}_G(s)\). Since \(\text{orb}_G(s) \subseteq S\), we also have \(\bigcup_{s \in S} \text{orb}_G(s) \subseteq S\). It remains to show that distinct orbits are actually disjoint. Suppose \(s, t \in S\) and \(x \in \text{orb}(s) \cap \text{orb}(t)\). Then \(\exists \alpha, \beta \in G \text{ s.t. } \alpha(s) = x = \beta(t)\).

Thus, \(\beta^{-1}\alpha(s) = \beta^{-1}(x) = t\). If \(y \in \text{orb}(t)\), then \(\exists \gamma \in G \text{ s.t. } y = \gamma(t) = \gamma(\beta^{-1}\alpha(s))\) so \(y \in \text{orb}(s)\). Thus, \(\text{orb}(t) \subseteq \text{orb}(s)\). Conversely, \(\alpha^{-1}\beta(t) = s\) implies that if \(y \in \text{orb}(s)\), then \(\exists \gamma \in G \text{ s.t. } y = s = \delta(\alpha^{-1}\beta(t)) \in \text{orb}(t)\), so \(\text{orb}(s) \subseteq \text{orb}(t)\). Thus, \(\text{orb}(s) = \text{orb}(t)\) and the set of orbits forms a partition of \(S\).

45. Let \(G = \{1, (1, 2, 3, 4), (1234), (1324), (1432), (3456), (5613), (14)(23), (24)(56)\}\)

(a) \(\text{stab}_G(1) = \{(1), (24)(56)\}\) \(\text{orb}_G(1) = \{1, 2, 3, 4\}\)

(b) \(\text{stab}_G(3) = \{(1), (24)(56)\}\) \(\text{orb}_G(3) = \{3, 4, 1, 2\}\)

(c) \(\text{stab}_G(5) = \{(1), (1234), (1324), (1432)\}\) \(\text{orb}_G(5) = \{5, 6\}\)

65. If \(G\) is a finite group with \(|G|<100\) and \(G\) has subgroups of orders 10 and 25, what is \(|G|\)?

By Lagrange's Thm, \(|G|\) must be a multiple of 10 and 25. We have \(\text{lcm}(10, 25) = 50\), so by HW 0.10, \(|G|\) is a multiple of 50. But since \(|G|<100\), we must have \(|G| = 50\).
2. Show that \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) has seven subgroups of order 2.

Note that every nonidentity element of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) has order 2 [the elements are \((1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\)], so each of these elements generates a subgroup of order 2, and there are seven of these. Every subgroup of order 2 is generated by an element of order 2, and since the only other element of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) is the identity of order 1, there are no additional subgroups of order 2.

3. Let \( G \) be a group with identity \( e_G \) and \( H \) a group with identity \( e_H \). Prove that \( G \cong G \oplus \{ e_H \} \) and \( H \cong \{ e_G \} \oplus H \).

**Proof:** Define \( \phi : G \to G \oplus \{ e_H \} \) by \( \phi(g) = (g, e_H) \). Let \( g_1, g_2 \in G \) and suppose \( \phi(g_1) = \phi(g_2) \). Then \((g_1, e_H) = (g_2, e_H) \Rightarrow g_1 = g_2 \), so \( \phi \) is injective. If \((g, e_H) \in G \oplus \{ e_H \}\), then \( \phi(g) = (g, e_H) \), so \( \phi \) is surjective. Let \( g_1, g_2 \in G \). Then \( \phi(g_1 g_2) = (g_1 g_2, e_H) = (g_1, e_H) \cdot (g_2, e_H) = \phi(g_1) \phi(g_2) \), so \( \phi \) preserves the group operations. Thus, \( \phi \) is an isomorphism.

Define \( \psi : H \to \{ e_G \} \oplus H \) by \( \psi(h) = (e_G, h) \). By symmetry with def. of \( \phi \), \( \psi \) is also an isomorphism.

4. Show that \( G \oplus H \) is abelian if and only if \( G \) and \( H \) are abelian. State the general case.

**Proof:** Suppose first that \( G \oplus H \) is abelian. Let \( g_1, g_2 \in G \) and \( h_1, h_2 \in H \). Then \((g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2) \Rightarrow (g_2, h_2)(g_1, h_1) = (g_2 g_1, h_2 h_1) \Rightarrow g_1 g_2 = g_2 g_1 \) and \( h_1 h_2 = h_2 h_1 \). Thus, \( G \) and \( H \) are abelian.

Conversely, suppose \( G \oplus H \) is abelian. Let \( g_1, g_2 \in G \), \( h_1, h_2 \in H \). Then \( g_1 g_2 = g_2 g_1 \) and \( h_1 h_2 = h_2 h_1 \) \Rightarrow \((g_1, h_1)(g_2, h_2) = (g_1 h_1)(g_2 h_2) \Rightarrow (g_1, h_1)(g_2, h_2) = (g_1, h_1)(g_2, h_2) \). Thus, \( G \oplus H \) is abelian.

**General case:** \( G_1 \oplus \ldots \oplus G_n \) is abelian if and only if \( G_i \) is abelian \( \forall i \in \{1, \ldots, n\} \).

5. Prove, by comparing orders of elements, that \( \mathbb{Z}_8 \oplus \mathbb{Z}_2 \neq \mathbb{Z}_4 \oplus \mathbb{Z}_4 \).

Note that \((5, 1) \in \mathbb{Z}_8 \oplus \mathbb{Z}_2 \) is an element of order 8, but there is no element of order 8 in \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \). If \( a, b \in \mathbb{Z}_4 \), then \( 4a = 4b = 0 \) by a corollary of Lagrange's Thm, so \( \phi((a, b)) = (4a, 4b) = (0, 0) \), which implies \(|(a, b)| \leq 4 \). Thus, no element of \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \) has order 8, so \( \mathbb{Z}_8 \oplus \mathbb{Z}_2 \neq \mathbb{Z}_4 \oplus \mathbb{Z}_4 \).

6. Prove that \( G_1 \oplus G_2 \cong G_2 \oplus G_1 \). State the general case.

**Proof:** Let \( \phi : G_1 \oplus G_2 \to G_2 \oplus G_1 \) be given by \( \phi(g_1, g_2) = (g_2, g_1) \) for \( g_1 \in G_1, g_2 \in G_2 \).

Let \( g_1, g_1' \in G_1, g_2, g_2' \in G_2 \) and suppose that \( \phi(g_1, g_2) = \phi(g_1', g_2') \). Then \((g_2, g_1) = (g_2', g_1') \Rightarrow g_2 = g_2' \Rightarrow g_1 = g_1' \). Thus \( \phi \) is injective.

If \((g_1, g_2) \in G_2 \oplus G_1 \), then \((g_1, g_2) \in G_1 \oplus G_2 \) and \( \phi(g_1, g_2) = (g_2, g_1) \), so \( \phi \) is surjective.

Let \( g_1, g_1' \in G_1, g_2, g_2' \in G_2 \). Then \( \phi(g_1, g_2) \phi(g_1', g_2') = (g_2, g_1)(g_2', g_1') = (g_2 g_2', g_1 g_1') = \phi(g_1 g_1', g_2 g_2') \). Thus \( \phi \) preserves group operations, so \( \phi \) is an isomorphism.

**General case:** \( G_1 \oplus \ldots \oplus G_n \cong G_n \oplus \ldots \oplus G_1 \).
8. Is \( \mathbb{Z}_3 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_{27} \)? Why?

No. Corollary 2 to Thm 8.2 states that \( \mathbb{Z}_m \oplus \mathbb{Z}_n \cong \mathbb{Z}_{mn} \) iff \( \gcd(m,n) = 1 \). Since \( \gcd(3,9) = 3 \neq 1 \), we cannot have \( \mathbb{Z}_3 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_{27} \).

9. Is \( \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{15} \)? Why?

Yes, by the Corollary referenced in #2, \( \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{15} \) since \( \gcd(3,5) = 1 \).

10. How many elements of order \( 9 \) does \( \mathbb{Z}_3 \oplus \mathbb{Z}_9 \) have?

Let \( (a,b) \in \mathbb{Z}_3 \oplus \mathbb{Z}_9 \). We know \( |\{(a,b)\}| = 9 \) if \( \text{lcm}(15,16) = 9 \). Now \( \text{lcm}(1,3,9) \) and \( \text{lcm}(1,3,9) \) by Lagrange's Thm. So \( |\{(a,b)\}| = 9 \Rightarrow (b|9 \Rightarrow b \text{ generates } \mathbb{Z}_9 \). Recall that \( \mathbb{Z}_9 \) has \( \phi(9) = 6 \) generators. There is no restriction on \( a \in \mathbb{Z}_3 \), so there are \( 3 \cdot 6 = 18 \) elements of order \( 9 \) in \( \mathbb{Z}_3 \oplus \mathbb{Z}_9 \).

16. Suppose that \( G_1 \cong G_2 \) and \( H_1 \cong H_2 \). Prove that \( G_1 \oplus H_1 \cong G_2 \oplus H_2 \) and state the general case.

**Proof** Since \( G_1 \cong G_2 \), there is an isomorphism \( \phi: G_1 \to G_2 \); similarly, \( H_1 \cong H_2 \), there is an isomorphism \( \psi: H_1 \to H_2 \). Define \( \alpha: G_1 \oplus H_1 \to G_2 \oplus H_2 \) by \( \alpha(g,h) = (\phi(g), \psi(h)) \) for \( g \in G_1 \), \( h \in H_1 \). Let \( g_1, g_2 \in G_1 \), \( h_1, h_2 \in H_1 \) and suppose that \( \alpha(g_1, h_1) = \alpha(g_2, h_2) \). Then 
\[
(\phi(g_1), \psi(h_1)) = (\phi(g_2), \psi(h_2)) \Rightarrow \phi(g_1) = \phi(g_2) \text{ and } \psi(h_1) = \psi(h_2) \] since \( \phi \) and \( \psi \) are injective, this implies \( g_1 = g_2 \), \( h_1 = h_2 \Rightarrow (g_1, h_1) = (g_2, h_2) \). Thus, \( \alpha \) is injective. Let \( g_2 \in G_2 \), \( h_2 \in H_2 \).

Since \( \phi \) is surjective, \( \exists g_1 \in G_1 \) st. \( \phi(g_1) = g_2 \); similarly, \( \exists h_1 \in H_1 \) st. \( \psi(h_1) = h_2 \). Then \( (g_1, h_1) \in G_1 \oplus H_1 \) and \( \alpha(g_1, h_1) = \alpha(g_2, h_2) \). Finally, let \( g, h \in G \) and \( h' \in H_1 \). Then since \( \phi, \psi \) are operation-preserving, we have \( \alpha((g,h)(g',h')) = \alpha((g_1,h_1)(g_2,h_2)) = \alpha((g_2,h_2) = \alpha((g_1,h_1)) = \alpha(g_1, h_1) = (\phi(g), \psi(h))) = (\phi(g_1), \psi(h_1)) = (\phi(g), \psi(h)) = \alpha((g,h)(g',h')) \). Thus, \( \alpha \) is also operation-preserving. 

**General case:** If \( G_1 \cong H_1, \ldots, G_n \cong H_n \), then \( G_1 \oplus \cdots \oplus G_n \cong H_1 \oplus \cdots \oplus H_n \).

7. If \( G \oplus H \) is cyclic, prove that \( G \) and \( H \) are cyclic. State the general case.

**Proof** Assume that \( G \oplus H \) is cyclic. Now \( |G \oplus H| = |G| \cdot |H| \) is divisible by \( |G| \cdot |H| \), so by the Fund. Thm of Cyclic Groups, there is a unique subgroup of \( G \oplus H \) of order \( |G| \cdot |H| \), and it is cyclic.

We know \( G \oplus \{e_H\} \cong G \oplus H \) and \( G \oplus \{e_H\} \oplus \{e_H\} \cong G \oplus H \), so it must be this unique subgroup. Therefore, \( G \oplus \{e_H\} \cong G \oplus H \). Then by #2, \( G \oplus \{e_H\} \cong G \), so \( G \) is also cyclic.

Replacing \( G \) with \( H \) and \( H \) with \( G \oplus \{e_H\} \) with \( e_G \), there is an isomorphism \( G \oplus \{e_H\} \cong H \). This argument yields that \( H \) is cyclic. 

**General case:** If \( G_1, \ldots, G_n \) is cyclic, then \( G \) is cyclic for \( G = G_1 \oplus \cdots \oplus G_n \).

8. In \( \mathbb{Z}_{40} \oplus \mathbb{Z}_{30} \), find two subgroups of order \( 12 \).

Since \( 6 \cdot 2 = 12 \), with \( 2 \cdot 20 \) and \( 6 \cdot 5 \), find a subgroup of \( \mathbb{Z}_{40} \) of order 2 and a subgroup of \( \mathbb{Z}_{30} \) of order 6 and take their external direct product: \( H = \langle \overline{10} \rangle \oplus \langle \overline{5} \rangle \).

Similarly, \( 3 \cdot 4 = 12 \), with \( 3 \cdot 10 \) and \( 4 \cdot 5 \), so \( K = \langle \overline{15} \rangle \oplus \langle \overline{5} \rangle \).
20. Find a subgroup of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) that is isomorphic to \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \).

Want a subgroup of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) of order 4 and a subgroup of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) of order 1. So we have

\[
\langle (\bar{3}, \bar{2}) \rangle \oplus \langle (\bar{2}, \bar{2}) \rangle \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \text{and} \quad \langle \bar{2} \rangle \oplus \langle \bar{2} \rangle \subseteq \mathbb{Z}_4 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_8 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_{20} \cong \mathbb{Z}_{36}
\]

Since \( \gcd(4, 9) = 1 \).

21. Let \( G \) and \( H \) be finite groups and \((g, h) \in G \oplus H\). State a necessary and sufficient condition

for \( \langle (g, h) \rangle = \langle g \rangle \oplus \langle h \rangle \).

Since we always have \( \langle (g, h) \rangle \subseteq \langle g \rangle \oplus \langle h \rangle \), we need a condition for \( |\langle g \rangle \oplus \langle h \rangle| = |\langle g, h \rangle| \).

We have \( 1 \langle (g, h) \rangle = 1 \langle g \rangle \langle h \rangle = \text{lcm}(|g|, |h|) \) and \( 1 \langle g \rangle \langle h \rangle = |g| \cdot |h| \). Then \( \text{lcm}(|g|, |h|) = |g| \cdot |h| \) if and only if \( \gcd(|g|, |h|) = 1 \).

22. Determine the number of elements of order 15 and the number of cyclic subgroups of order 15 in \( \mathbb{Z}_{30} \oplus \mathbb{Z}_{20} \).

Want \( (\bar{a}, \bar{b}) \in \mathbb{Z}_{30} \oplus \mathbb{Z}_{20} \) s.t. \( \text{lcm}(|\bar{a}|, |\bar{b}|) = 15 \). Then we must have \( |\bar{a}| = 3, |\bar{b}| = 5, |\bar{a}| = 15, |\bar{b}| = 1, \) or \( |\bar{a}| = 15, |\bar{b}| = 1 \). There are \( \phi(3) = 2 \) elements of order 3 in \( \mathbb{Z}_{30} \) and \( \phi(5) = 4 \) elements of order 5 in \( \mathbb{Z}_{20} \), so the first case yields \( 2 \cdot 4 = 8 \) elements. There are \( \phi(15) = 8 \) elements of order 15 in \( \mathbb{Z}_{30} \) and only 1 element of order 1 in \( \mathbb{Z}_{20} \) (the identity), so the third case yields \( 8 \cdot 1 = 8 \) elements. Thus, there are \( 48 \) elements of order 15 in \( \mathbb{Z}_{30} \oplus \mathbb{Z}_{20} \).

Since each cyclic subgroup of order 15 contains \( \phi(15) = 8 \) elements of order 15, and all such elements must lie in only one cyclic subgroup, there are \( \frac{48}{8} = 6 \) cyclic subgroups of order 15 in \( \mathbb{Z}_{30} \oplus \mathbb{Z}_{20} \).

23. What is the order of any nonidentity element of \( \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \)? Generalize.

Let \( (\bar{a}, \bar{b}, \bar{c}) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \) and \( \langle \bar{0}, \bar{0}, \bar{0} \rangle \neq \langle \bar{a}, \bar{b}, \bar{c} \rangle \). By a Corollary to Lagrange's Thm,

\[3\bar{a} + 3\bar{b} + 3\bar{c} = \bar{0}, \text{ so } 1 \bar{a} \in \langle 1, 1, 1 \rangle, 1 \bar{b} \in \langle 1, 1, 1 \rangle, 1 \bar{c} \in \langle 1, 1, 1 \rangle. \]

Since \( 1 \langle \bar{a}, \bar{b}, \bar{c} \rangle = \text{lcm}(|1\bar{a}|, |1\bar{b}|, |1\bar{c}|) \neq 1 \), we must have \( |1\bar{a}| = 3 \) for some \( j \in \{a, b, c\} \). Thus, \( 1 \langle \bar{a}, \bar{5}, \bar{c} \rangle = \text{lcm}(|1\bar{a}|, |1\bar{5}|, |1\bar{c}|) = 3 \).

Generalization: The order of any nonidentity element of \( \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p \), where \( p \) is a prime, is \( p \).

Let \( G \) be a group and \( H = \{ (g, g) \mid g \in G \} \). Show that \( H \leq G \oplus G \). When \( G = \langle \mathbb{R}, + \rangle \), describe \( G \oplus G \) and \( H \) geometrically.

Proof. Since \( e_G \in G \), \( (e_G, e_G) \in H \), so \( H \) is nonempty. Let \( g, h \in G \) and \( (g, g), (h, h) \in H \). Then \( (g, g)(h, h) = (gh, gh) \in H \) since \( gh \in G \). Finally, let \( g \in G \) and \( (g, g) \in H \). Then

\[
(g, g)^{-1} = (g^{-1}, g^{-1}) \in H \since g^{-1} \in G. \text{ Thus, } H \triangleleft G \oplus G. \quad \Box
\]

When \( G = \langle \mathbb{R}, + \rangle \), \( G \oplus G \) is the Cartesian plane and \( H \) is the line \( y = x \).

There are \( \phi(15) = 8 \) elements of order 15 in \( \mathbb{Z}_{30} \) and \( \phi(5) = 4 \) elements of order 5 in \( \mathbb{Z}_{20} \), so the second case yields \( 8 \cdot 4 = 32 \) elements.
28. Find a subgroup of \( \mathbb{Z}_4 \oplus \mathbb{Z}_2 \) that is not of the form \( H \oplus K \), where \( H \leq \mathbb{Z}_4 \) and \( K \leq \mathbb{Z}_2 \).

Consider the subgroup \( \langle (1, 1) \rangle = \{ (0, 0), (1, 1), (3, 1), (3, 1) \} \). If it had the form \( H \oplus K \), then \( H = \{ 0, 1, 3 \} = \mathbb{Z}_4 \) and \( K = \{ 0, 1 \} = \mathbb{Z}_2 \); but certainly \( \langle (1, 1) \rangle \not\cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \) since \( |\langle (1, 1) \rangle| = 4 \neq 2 = |\mathbb{Z}_4 \oplus \mathbb{Z}_2| \).

52. Is \( \mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \cong \mathbb{Z}_{60} \oplus \mathbb{Z}_{6} \oplus \mathbb{Z}_2 \)?

We have \( \mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_5) \oplus (\mathbb{Z}_4 \oplus \mathbb{Z}_3) \oplus \mathbb{Z}_6 \)

\[ \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3) \oplus \mathbb{Z}_6 \]

\[ \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_6 \cong \mathbb{Z}_{60} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \) (by 11.7), so yes.

53. Is \( \mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \cong \mathbb{Z}_{15} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12} \)?

As in 15.2, \( \mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_5) \oplus (\mathbb{Z}_4 \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_3) \)

\[ \cong (\mathbb{Z}_1 \oplus \mathbb{Z}_5) \oplus (\mathbb{Z}_1 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3) \] (by 11.7)

\[ \cong \mathbb{Z}_{15} \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_3) \neq \mathbb{Z}_{15} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12} \]

since \( \mathbb{Z}_2 \oplus \mathbb{Z}_3 \neq \mathbb{Z}_4 \). So no.